PARABOLIC PDE's

Simple Explicit Method:

Given PDE: $u_{xx} = u_t$ in non-dimensional form $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$

$u(x,0) = f(x) \quad 0 \le x \le 1$ Initial Condition at $t = 0$

$u(0,t) = g(t) \quad t > 0$

$u(1,t) = h(t) \quad t > 0$

} Boundary Conditions

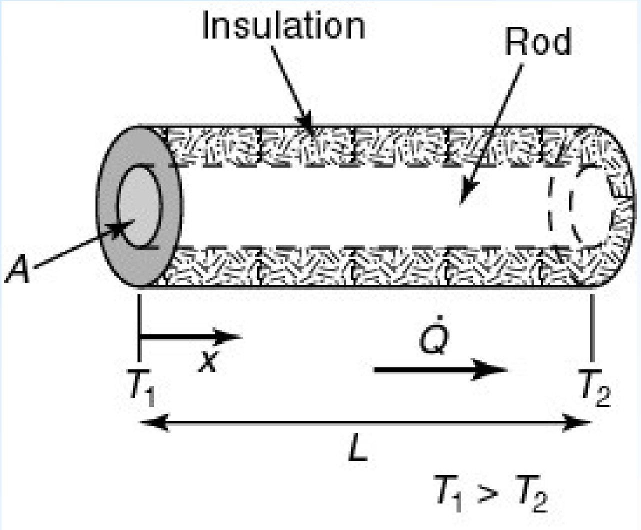
Corresponding finite-difference equation:

$$u_{i\,,\,j+1} = r\, u_{i-1\,,\,j} + (1 - 2\,r)\, u_{i\,,\,j} + r\, u_{i+1\,,\,j}$$

where $r = \dfrac{\Delta t}{(\Delta x)^2} = \dfrac{k}{h^2}$

Stability Condition: $r \le \dfrac{1}{2}$?

Example of a Parabolic Equation



A thin rod of length L extending between two plates is well insulated.

The initial temperature distribution at t = 0 is f(x).

Suddenly two end plates are subjected to g(t) and h(t).

PDE: $$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T(x,t)}{\partial t}$$

α: thermal diffusivity in m^2/s
t: time in s

IC: $T(x,0) = f(x)$ at $t = 0$ and $0 \le x \le L$

BC's: $T(0,t) = g(t)$ at $x = 0$ and $t > 0$

$T(L,t) = h(t)$ at $x = L$ and $t > 0$

Let's make these non-dimensional

WHY?

Distinguish between dimensionless and non-dimensional

No dimension at all — Ratio of the same dimensions

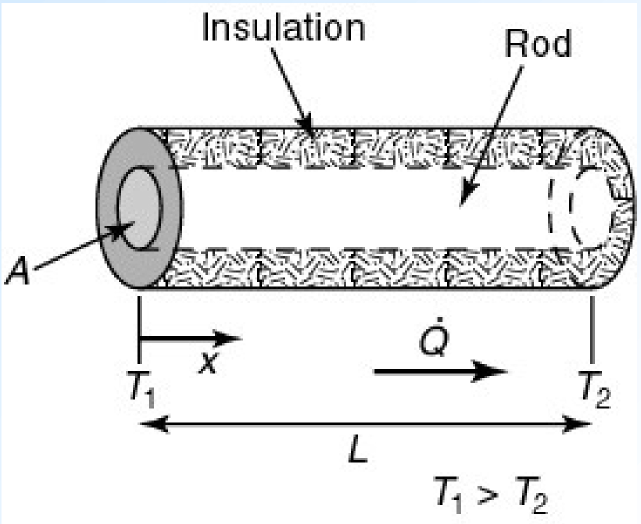
Why Non-dimensionalasion? - Dimensional analysis - Scaling

Non-dimensionalisation has several important uses:

1. It identifies the dimensionless groups (ratios of dimensional parameters) which control the solution behavior. Remember Re, Pr, Nu, Gr, St, Pe, etc.
2. Terms in the equations are now dimensionless and so allows comparison of their sizes. This allows the identification of the important (i.e. dominant) terms in the equations and their interaction in different regimes, giving insight into the structure of solutions and the dominant physical mechanisms at work.

2. In particular, negligible terms can be identified leading to simplification in many circumstances.
3. It allows estimates of the effects of additional features to the original model through the new dimensional group(s) associated with the additional term(s). This allows measurement of the effect of the physical feature(s) in the model.
4. It can reduce the number of parameters occurring in the problem by forming the non-dimensional parameters or dimensionless groups.
5. It facilitates the numerical solution of the mathematics (PDE) and its interpretation in the physical realm of the problem.
6. Others …

Non-dimensional Parabolic Equation



Define non-dimensional parameters:

$$x^+ = \frac{x}{L} \quad \text{and} \quad t^+ = \frac{\alpha}{L^2}\, t$$

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial x^+}\frac{dx^+}{dx} = \frac{\partial T}{\partial x^+}\frac{1}{L} \quad \Rightarrow \quad \frac{\partial^2 T}{\partial x^2} = \frac{\partial^2 T}{\partial x^{+2}}\frac{1}{L^2}$$

$$\frac{\partial T}{\partial t} = \frac{\partial T}{\partial t^+}\frac{dt^+}{dt} = \frac{\partial T}{\partial t^+}\frac{\alpha}{L^2}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

PDE:

$$\frac{\partial^2 T}{\partial x^{+2}} = \frac{\partial T}{\partial t^+}$$

$$T_{xx} = T_t$$

IC: $T(x^+,0) = f(x^+)$ at $t^+ = 0$ and $0 \le x^+ \le 1$

BC's: $T(0,t^+) = g(t^+)$ at $x^+ = 0$ and $t^+ > 0$

$T(1,t^+) = h(t^+)$ at $x^+ = 1$ and $t^+ > 0$

PDE: $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$ or $u_{xx} = u_t$, x and t are non-dimensional

IC: $u(x,0) = f(x)$ at $t = 0$ and $0 \le x \le 1$

BC's: $u(0,t) = g(t)$ at $x = 0$ and $t > 0$

$u(1,t) = h(t)$ at $x = 1$ and $t > 0$

Replace the space derivative by central difference and time derivative by forward difference **(WHY?)**:

$$\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} \cong \frac{u_{i,j+1} - u_{i,j}}{k}$$

where $h = \Delta x$ and $k = \Delta t$

Rearrange: $u_{i,j+1} = r\,u_{i-1,j} + (1 - 2r)\,u_{i,j} + r\,u_{i+1,j}$ where $r = \dfrac{\Delta t}{(\Delta x)^2} = \dfrac{k}{h^2}$

Remark on the stability condition:

The order of the simple explicit formula (truncation error) is

$O(k^2 + k\,h^2)$ or $O(k^2) + O(k\,h^2)$

These terms evaluated at i,j are:

$$\frac{k^2}{2}\frac{\partial^2 u}{\partial t^2} - \frac{k\,h^2}{12}\frac{\partial^4 u}{\partial x^4} = \frac{k}{2}\left[k\frac{\partial^2 u}{\partial t^2} - \frac{h^2}{6}\frac{\partial^4 u}{\partial x^4}\right]$$

Since $u_t = u_{xx}$ => $u_{tt} = u_{xxt} = u_{txx} = u_{xxxx}$ So,

$$\frac{k^2}{2}\frac{\partial^2 u}{\partial t^2} - \frac{k\,h^2}{12}\frac{\partial^4 u}{\partial x^4} = \frac{k}{2}\left[k\frac{\partial^2 u}{\partial t^2} - \frac{h^2}{6}\frac{\partial^4 u}{\partial x^4}\right] = \frac{k}{2}\left[k - \frac{h^2}{6}\right]\frac{\partial^4 u}{\partial x^4}$$

$$\frac{k^2}{2}\frac{\partial^2 u}{\partial t^2} - \frac{k\,h^2}{12}\frac{\partial^4 u}{\partial x^4} = \frac{k}{2}\left[k\frac{\partial^2 u}{\partial t^2} - \frac{h^2}{6}\frac{\partial^4 u}{\partial x^4}\right] = \frac{k}{2}\left[k - \frac{h^2}{6}\right]\frac{\partial^4 u}{\partial x^4}$$

This shows that if $k / h^2 = 1 / 6$, then the truncation error will be $O(k^3)$ which is the same as $O(h^6)$. So, if we choose non-dimensional **amplification factor** as $r = 1 / 6$, the solution of the difference equation approaches the real solution with special rapidity.

Example Problem: $\dfrac{\partial^2 u(x,t)}{\partial x^2} = \dfrac{\partial u(x,t)}{\partial t}$, $0 < x < 1$, $t > 0$

Initial Condition: $u(x,0) = x(1-x)$, $0 \le x \le 1$, $t = 0$

Boundary Conditions: $u(0,t) = u(1,t) = 0$, $t > 0$

Exact solution using **separation of variables**:

$$u(x,t) = \frac{8}{\pi^3} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi x}{(2n+1)^3} e^{-(2n+1)^2 \pi^2 t}$$

Numerical Solution: Set $h = 0.2$, $k = 0.01$, $r = 0.25$

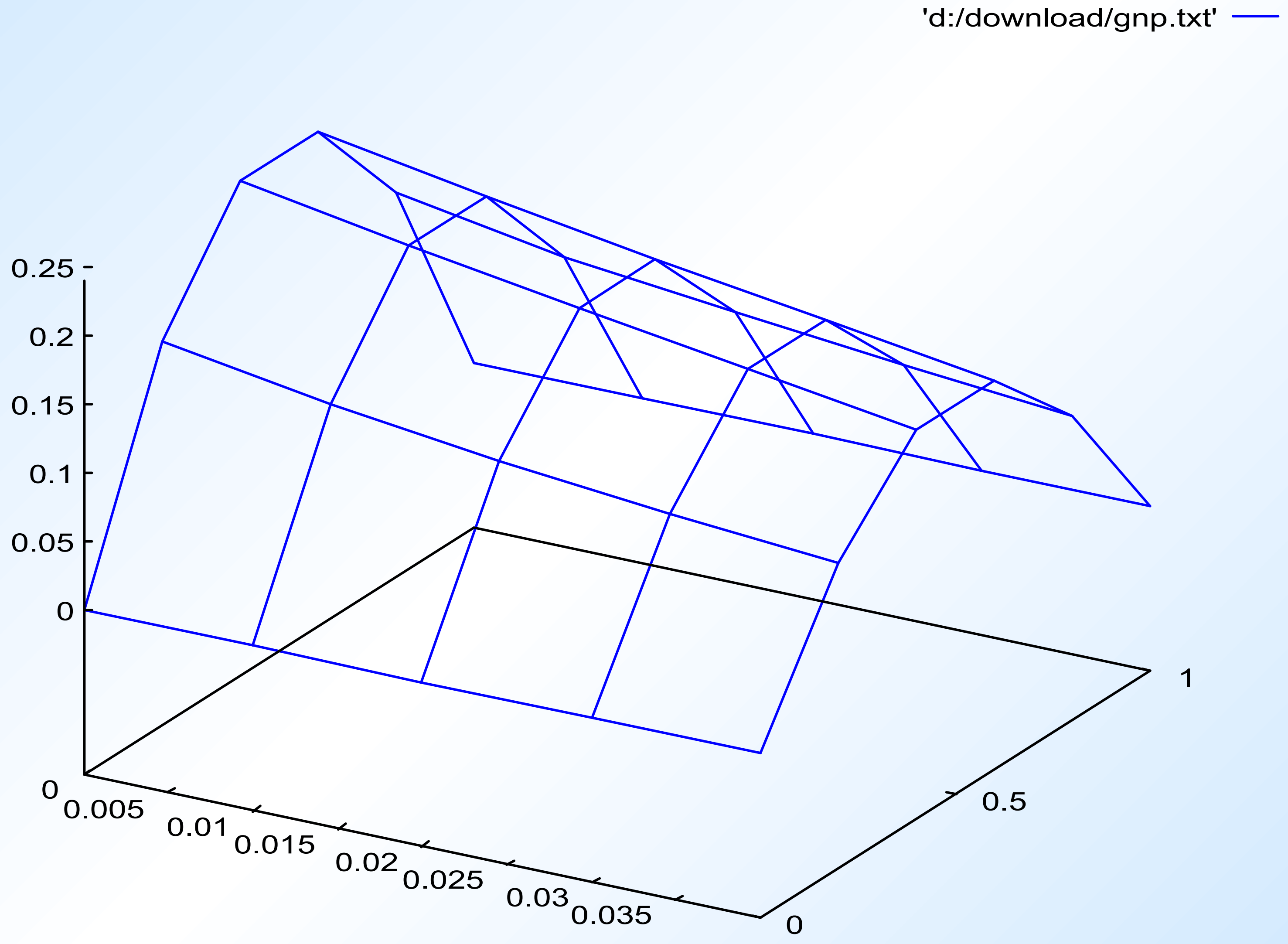
The finite difference equation becomes:

$$u_{i,\,j+1} = \frac{1}{4}\left(u_{i-1,\,j} + 2\,u_{i,\,j} + u_{i+1,\,j}\right)$$

Initial Condition: $u_{i,\,0} = x_i\left(x_i - 1\right)$ for $i = 0, 1, ..., 5$

Boundary Condition: $u_{0,\,j} = u_{5,\,j} = 0$ for $j > 0$

j	$u_{0,j}$	$u_{1,j}$	$u_{2,j}$	$u_{3,j}$	$u_{4,j}$	$u_{5,j}$
0	0	0.14	0.22	0.22	0.14	0
1	0	0.125	0.2	0.2	0.125	0
2	0	0.1125	0.1813	0.1813	0.1125	0


'd:/download/gnp.txt'
0.25
0.2
0.15
0.1
0.05
0
0
0.005
0.01
0.015
0.02
0.025
0.03
0.035
0
0.5
1

STABILITY

Let $L\{U\} = 0$ given PDE

$F\{u_{i,j}\} = 0$ Corresponding finite difference scheme

The finite difference scheme is said to be **convergent** (to the PDE) if $u_{i,j}$ tends to the exact solution $U(x_i,y_j)$ (or $U_{i,j}$) as h and k tend to zero.

The difference $d_{i,j} = U_{i,j} - (u_{i,j})^*$ is called the *cumulative truncation (or discretization) error*. $(u_{i,j})^*$ is the exact solution of the difference equation.

$d_{i,j}$ depends on grid sizes, h and k, as well as the number of terms used in the truncated series to approximate each partial derivative.

If the exact finite-difference solution $(u_{i,j})^*$ is replaced by the exact solution of the PDE, $U_{i,j}$ at the grid point, $P_{i,j}$, then the value of $F\{U_{i,j}\}$ is called the *local truncation error* at $P_{i,j}$.

The finite difference scheme and the PDE are said to be **consistent** if $F\{U_{i,j}\}$ tends to zero as h and k tend to zero.

Define $r_{i,j} = (u_{i,j})^* - u_{i,j}$

where $(u_{i,j})^*$ is the exact solution of the difference equation

$u_{i,j}$ is the actual solution of the difference equation

$r_{i,j}$ is called the round-off error

The total error is $$U_{i,j} - u_{i,j} = U_{i,j} - (u_{i,j})^* + (u_{i,j})^* - u_{i,j}$$

$$U_{i,j} - u_{i,j} = d_{i,j} + r_{i,j}$$

Therefore, $d_{i,j}$ is bounded when $u_{i,j}$ is bounded.

The finite difference algorithm is said to be **stable** if the round-off errors are sufficiently small for all i as $j \to \infty$ i.e., the growth of $r_{i,j}$ can be controlled.

Note that $r_{i,j}$ depends on type of computer used, the computational process, and on the finite-difference equation itself.

STABILITY CRITERION

Lax's Equivalence Theorem:

Given a **properly-posed**, **linear**, **initial-value** problem and a finite-difference approximation to it that satisfies the consistency criterion, stability is the necessary and sufficient condition for convergence.

Note that this statement needs to be interpreted

A general, second-order, linear, parabolic PDE looks like:

$$\frac{\partial}{\partial x}\left(a(x,t)\ \frac{\partial u}{\partial x}\right) + \mathrm{b}(x,t)\ \frac{\partial u}{\partial x} + \mathrm{c}(x,t)\ u = \mathrm{d}(x,t)\ \frac{\partial u}{\partial t}$$

Peter David Lax

Hungarian (American) Mathematician

1926 -

Let $u_{i,j}$ be the numerical (finite difference) solution, and $U_{i,j}$ be the exact solution of the PDE. The error is defined as

$$U_{i,j} - u_{i,j} = E_{i,j} = E(x_i, t_j)$$

The exact solution U has the form

$$U(x,t) = e^{\sqrt{-1}\,\lambda\, x}\; e^{\alpha\, t}$$

$$U(x_i,t_j) = U_{i,j} = e^{\sqrt{-1}\,\lambda\, i\, h}\; e^{\alpha\, j\, k}$$

where λ is a real number and α can be a complex number

The error $E_{i,j}$ must therefore have the same form: $E_{i,j} = e^{\sqrt{-1}\,\lambda\, i\, h}\; e^{\alpha\, j\, k}$

This error $E_{i,j} = e^{\sqrt{-1}\,\lambda\, i\, h}\; e^{\alpha\, j\, k}$

must stay finite (for stability) as t (or j) goes to infinity for all α

The **Von Neumann criterion for stability** is $\left|e^{\alpha\, k}\right| \leq 1$ α can be complex

Example: Determine the stability condition for the simple explicit method

$$u_{i\,,\,j+1} = r\, u_{i-1\,,\,j} + (1 - 2\,r)\, u_{i\,,\,j} + r\, u_{i+1\,,\,j}$$

where $r = \dfrac{\Delta\, t}{(\Delta\, x)^2} = \dfrac{k}{h^2}$

$$r \leq \frac{1}{2\,\sin^2\left(\dfrac{\lambda\, h}{2}\right)}$$

John von Neumann

Hungarian (American) Mathematician

1903 - 1957

Substitute $u_{i,j} = e^{\sqrt{-1}\,\lambda\, i\, h}\; e^{\alpha\, j\, k}$ into the finite difference equation

$$u_{i\,,\,j+1} = r\, u_{i-1\,,\,j} + (1 - 2\, r)\, u_{i\,,\,j} + r\, u_{i+1\,,\,j}$$

$$e^{\sqrt{-1}\,\lambda\, i\, h}\; e^{\alpha\,(j+1)\,k} = r\, e^{\sqrt{-1}\,\lambda\,(i-1)\,h}\; e^{\alpha\, j\, k} + (1 - 2\, r)\, e^{\sqrt{-1}\,\lambda\, i\, h}\; e^{\alpha\, j\, k} + r\, e^{\sqrt{-1}\,\lambda\,(i+1)\,h}\; e^{\alpha\, j\, k}$$

Cancel the same terms from both sides of the equality

$$e^{\alpha\, k} = 1 - 2\, r + r\left(e^{-\sqrt{-1}\,\lambda\, h} + e^{\sqrt{-1}\,\lambda\, h}\right)$$

$$\left|e^{\alpha\, k}\right| = \left|1 - 4\, r \sin^2\left(\frac{\lambda\, h}{2}\right)\right| \leq 1 \quad \Rightarrow \quad \boxed{r \leq \frac{1}{2 \sin^2\left(\frac{\lambda\, h}{2}\right)}}$$

Given PDE: $U_{xx} = U_t$ in non-dimensional form $$\frac{\partial^2 U}{\partial x^2} = \frac{\partial U}{\partial t}$$

Corresponding finite-difference equation (simple explicit method):

$$u_{i\,,\,j+1} = r\,u_{i-1\,,\,j} + (1 - 2\,r)\,u_{i\,,\,j} + r\,u_{i+1\,,\,j}$$ where $$r = \frac{\Delta t}{(\Delta x)^2} = \frac{k}{h^2}$$

Note that there are two time levels, j and j+1

The finite difference scheme and the PDE are said to be **consistent** if $F\{U_{i,j}\}$ tends to zero as h and k tend to zero.

The finite difference scheme is said to be **convergent** (to the PDE) if $u_{i,j}$ tends to the exact solution $U(x_i,y_j)$ (or $U_{i,j}$) as h and k tend to zero.

LAX's equivalence therom:

Given a **properly-posed**, **linear**, **initial-value** problem and a finite-difference approximation to it that satisfies the consistency criterion, stability is the necessary and sufficient condition for convergence.

The finite difference algorithm is said to be **stable** if the round-off errors are sufficiently small for all i as $j \rightarrow \infty$ i.e., the growth of $r_{i,j}$ can be controlled.

Form of the exact solution, or the error, is: $E_{i,j} = e^{\sqrt{-1}\,\lambda\, i\, h}\; e^{\alpha\, j\, k}$

The **Von Neumann criterion for stability** is: $\left| e^{\alpha\, k} \right| \leq 1$ α can be complex

Note the following:

1. The method applies only if the coefficients of the linear difference equation are constant. For variable coefficients, the method can still be applied locally and it might be expected that the finite-difference solution will be stable if the Von Neumann condition, derived as though the coefficients were constant, is satisfied at every point in the solution field. There is much numerical evidence to support this.

2. For two-time-level difference schemes with one dependent variable and any number of independent variables, the Von Neumann condition is sufficient as well as necessary for stability. Otherwise, the condition is only necessary.

3. Boundary conditions are neglected by the Von Neumann method which applies, in theory, to only pure initial-value problems with periodic initial data. It does, however, provide necessary condition for stability of constant-coefficient problems regardless of the boundary conditions.

IMPLICIT METHODS – implicit in time

An implicit method is the one in which two or more unknown values in the j+1[th] row are specified in terms of the j[th] row (and j-1, j-2, etc., if necessary) by a single application of the expression.

One simple implicit method is suggested by **O'Brien**, by approximaing the second-order derivative, u_{xx}, in the j+1[th] row insread of the j[th] row

PDE: $$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad \text{or} \quad u_{xx} = u_t \quad , \quad x \text{ and } t \text{ are non-dimensional}$$

$$\frac{u_{i-1,j+1} - 2\,u_{i,j+1} + u_{i+1,j+1}}{h^2} \cong \frac{u_{i,j+1} - u_{i,j}}{k} \qquad \text{where } h = \Delta x \text{ and } k = \Delta t$$

$$u_{i\,,\,j} = -\,r\,u_{i-1\,,\,j+1} + (1 + 2\,r)\,u_{i\,,\,j+1} - r\,u_{i+1\,,\,j+1} \qquad \text{where} \quad r = \frac{\Delta t}{(\Delta x)^2} = \frac{k}{h^2}$$

Matthew O'Brien

Irish Mathematician

1814 - 1855

Richardson's Explicit Finite-Difference Scheme

$$\frac{u_{i-1,j} - 2\, u_{i,j} + u_{i+1,j}}{h^2} \cong \frac{u_{i,j+1} - u_{i,j-1}}{2\, k}$$

Both derivatives are replaced by central differences.

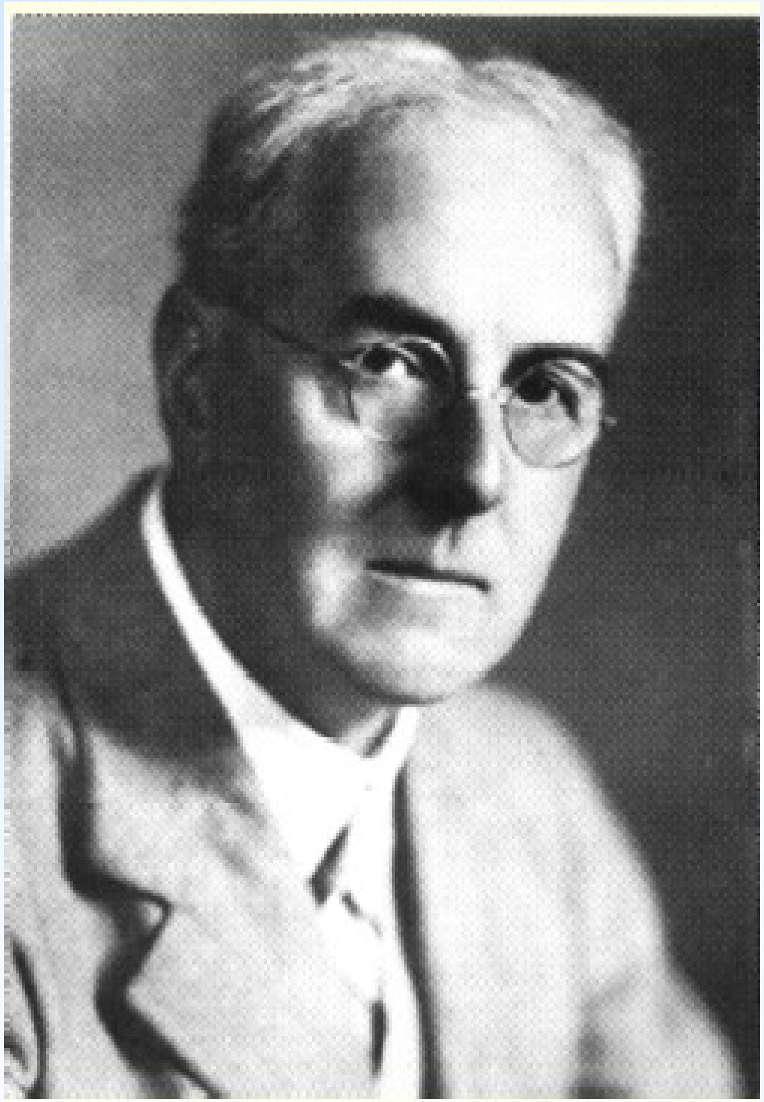
This is an attempt to improve the local truncation errors of the approximation for u_t.

This is also called overlapping-steps method.

Note that it is a three-time-level formula

However, such a scheme is unstable for all values of r, or it is unconditionally unstable.

Prove that it is the case.



Lewis Fry Richardson

British Mathematician

1881 - 1953

Du Fort-Frankel Explicit Algorithm

Replace $u_{i,j}$ with $\dfrac{u_{i,j+1} - u_{i,j-1}}{2}$

$$\frac{u_{i-1,j} - u_{i,j+1} - u_{i,j-1} + u_{i+1,j}}{h^2} \cong \frac{u_{i,j+1} - u_{i,j-1}}{2\,k}$$

This was proposed for linear diffusion equations with periodic boundary conditions by E.C. Du Fort and S.P. Frankel in 1953.

This is again a three-time-level formula, and it can be shown that **it is stable for all values of r**. (Prove this.)

However, it has the disadvantage that it requires a special starting procedure since one line of values (time step j = 1), in addition to the initial line (time step j = 0), must be known before the formula can be appllied.

Stanley Phillips Frankel

American computer scientist

1919 - 1978

Crank-Nicolson Method

Replace $u_{i,j}$ with $\dfrac{u_{i,j+1} - u_{i,j-1}}{2}$

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{1}{h^2}\left[\frac{u_{i-1,j+1} - 2\,u_{i,j+1} + u_{i+1,j+1}}{2} + \frac{u_{i-1,j} - 2\,u_{i,j} + u_{i+1,j}}{2}\right]$$

This is again a three-time-level formula, and it can be shown that **it is stable for all values of r**. (Prove this.)

$$u_{i-1,j+1} - 2\left(1 + \frac{1}{r}\right)u_{i,j+1} + u_{i+1,j+1} = -\left[u_{i-1,j} - 2\left(1 - \frac{1}{r}\right)u_{i,j} + u_{i+1,j}\right]$$

John Crank

British Mathematician

1916 - 2006

Phyllis (Lockett) Nicolson

British Mathematician

1917 - 1968

IMPLICIT METHODS

$$- r \lambda u_{i-1,j+1} + (1 + 2r\lambda) u_{i,j+1} - r \lambda u_{i+1,j+1}$$

$$= r(1-\lambda) u_{i-1,j} + [1 - 2r(1-\lambda)] u_{i,j} + r(1-\lambda) u_{i+1,j}$$

If $\lambda = 0$ Explicit relation

$\lambda = 1$ O'Brien et. al. formula

$\lambda = 1/2$ Crank-Nicolson formula

Example: Tri-diagonal coefficient matrix, stable for all r?

Matrix stability analysis !

Example

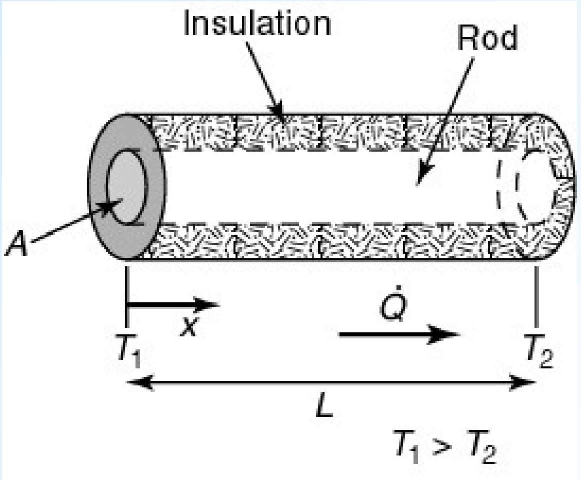
PDE: $u_{xx} = u_t$

IC: $u(x,0) = 1$ at $t = 0$

BC's: $u(0,t) = 0$ and $\left.\dfrac{\partial u}{\partial x}\right|_{x=1,t>0} = 1$

Let $h = 1/3$, $k = 1/6$ so $r = k/h^2 = 3/2$ and $\lambda = 2/3$

Example



PDE: $u_{xx} = u_t$ Heat equation

IC: $u(x,0) = \sin(\pi x)$, $t = 0$, $0 \le x \le 1$

BC's: $u(0,t) = u(1,t) = 0$, $t > 0$

Exact solution: $u(x,t) = e^{-\pi^2 t} \sin(\pi x)$

Let $h = 0.2$, $k = 0.05$ so $r = k/h^2 = 1.25$ and $\lambda = 1/2$ (Crank-Nicolson)

FDE: $u_{i-1,j+1} - 3.6\, u_{i,j+1} + u_{i+1,j+1} = -\left[u_{i-1,j} - 0.4\, u_{i,j} + u_{i+1,j}\right]$ $i = 1, 2, 3, 4$, $j > 0$

Initial condition (at t = 0, or j = 0):

$$u_{i,0} = \sin\left(\pi\, x_i\right) = \sin\left(\pi\, i\, h\right) = \sin\left(0.2\, \pi\, i\right) \quad , \quad i = 0, 1, ..., 5$$

Initial condition (at t = 0, or j = 0):

$$u_{i,0} = \sin\left(\pi\, x_i\right) = \sin\left(\pi\, i\, h\right) = \sin\left(0.2\, \pi\, i\right) \quad , \quad i = 0, 1, ..., 5$$

$u_{0,0}$ =	0.0000000
$u_{1,0}$ =	0.5877853
$u_{2,0}$ =	0.9510565
$u_{3,0}$ =	0.9510565
$u_{4,0}$ =	0.5877853
$u_{5,0}$ =	0.0000000

For higher accuracy, use a lot more number of internal points, not 4, but 40, or 400, ...

Boundary conditions: $u_{0,j} = u_{5,j} = 0 \quad , \quad j > 0$

FDE: $u_{i-1,j+1} - 3.6\, u_{i,j+1} + u_{i+1,j+1} = -\left[u_{i-1,j} - 0.4\, u_{i,j} + u_{i+1,j}\right] \qquad i = 1, 2, 3, 4 \quad , \quad j > 0$

For i = 1 $u_{0,1} - 3.6\, u_{1,1} + u_{2,1} = -\left[u_{0,0} - 0.4\, u_{1,0} + u_{2,0}\right] = -0.71594$

For i = 2 $u_{1,1} - 3.6\, u_{2,1} + u_{3,1} = -\left[u_{1,0} - 0.4\, u_{2,0} + u_{3,0}\right] = -1.15842$

For i = 3 $u_{2,1} - 3.6\, u_{3,1} + u_{4,1} = -\left[u_{2,0} - 0.4\, u_{3,0} + u_{4,0}\right] = -1.15842$

For i = 4 $u_{3,1} - 3.6\, u_{4,1} + u_{5,1} = -\left[u_{3,0} - 0.4\, u_{4,0} + u_{5,0}\right] = -0.71594$

In matrix notation:

$$\underbrace{\begin{pmatrix} -3.6 & 1 & 0 & 0 \\ 1 & -3.6 & 1 & 0 \\ 0 & 1 & -3.6 & 1 \\ 0 & 0 & 1 & -3.6 \end{pmatrix}}_{J} \begin{pmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ u_{4,1} \end{pmatrix} = \begin{pmatrix} -0.71594 \\ -1.15842 \\ -1.15842 \\ -0.71594 \end{pmatrix}$$

$$\begin{pmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ u_{4,1} \end{pmatrix} = J^{-1} \begin{pmatrix} -0.71594 \\ -1.15842 \\ -1.15842 \\ -0.71594 \end{pmatrix} = \begin{pmatrix} \ldots \\ \ldots \\ \ldots \\ .. \end{pmatrix}$$

J matrix does not have any i's or j's. Therefore, it may be inverted once and for all, and used for the rest of the calculations.

$$\begin{pmatrix} u_{1,2} \\ u_{2,2} \\ u_{3,2} \\ u_{4,2} \end{pmatrix} = J^{-1} \begin{pmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ u_{4,1} \end{pmatrix} = \begin{pmatrix} \ldots \\ \ldots \\ \ldots \\ .. \end{pmatrix} \qquad \begin{pmatrix} u_{1,3} \\ u_{2,3} \\ u_{3,3} \\ u_{4,3} \end{pmatrix} = J^{-1} \begin{pmatrix} u_{1,2} \\ u_{2,2} \\ u_{3,2} \\ u_{4,2} \end{pmatrix} = \begin{pmatrix} \ldots \\ \ldots \\ \ldots \\ .. \end{pmatrix}$$

How would you estimate the errors?

More on Stability

PDE: $u_{xx} = u_t$

One condition on time (IC) and two conditions on space (BC's)

Simple Explicit Method: $u_{i\,,\,j+1} = r\,u_{i-1\,,\,j} + (1 - 2\,r)\,u_{i\,,\,j} + r\,u_{i+1\,,\,j}$, $r > 0.5$

Crank-Nicolson Method:

$$u_{i-1,j+1} - 2\left(1 + \frac{1}{r}\right)u_{i,j+1} + u_{i+1,j+1} = -\left[u_{i-1,j} - 2\left(1 - \frac{1}{r}\right)u_{i,j} + u_{i+1,j}\right] \quad , \quad \text{all } r$$

Given consistency (between the PDE and the FDE)

Prove stability => convergence of FDE sol'n to PDE sol'n

} Lax's equivalence theorem

How do you prove stability?

Given consistency (between the PDE and the FDE)

Prove stability => convergence of FDE sol'n to PDE sol'n

} Lax's equivalence theorem

How do you prove stability?

Substitute $u_{i,j} = e^{\sqrt{-1}\,\lambda\, i\, h}\; e^{\alpha\, j\, k}$ into the finite difference equation (FDE)

Von Neumann stability criterion: $\left| e^{\alpha\, k} \right| \leq 1$ α can be complex

Remember the restrictions:

1. The method applies only if the coefficients of the linear difference equation are constant.

$$\frac{\partial}{\partial x}\left(a(x,t)\ \frac{\partial u}{\partial x} \right) + \mathrm{b}(x,t)\ \frac{\partial u}{\partial x} + \mathrm{c}(x,t)\ u = \mathrm{d}(x,t)\ \frac{\partial u}{\partial t}$$

For variable coefficients, the method can still be applied locally and it might be expected that the finite-difference solution will be stable if the Von Neumann condition, derived as though the coefficients were constant, is satisfied at every point in the solution field. There is much numerical evidence to support this.

2. For two-time-level difference schemes with one dependent variable and any number of independent variables, the Von Neumann condition is sufficient as well as necessary for stability. Otherwise, the condition is only necessary.

$$u_{i-1,j+1} - 2\left(1 + \frac{1}{r}\right)u_{i,j+1} + u_{i+1,j+1} = -\left[u_{i-1,j} - 2\left(1 - \frac{1}{r}\right)u_{i,j} + u_{i+1,j}\right]$$

3. Boundary conditions are neglected by the Von Neumann method which applies, in theory, to only pure initial-value problems with periodic initial data. It does, however, provide necessary condition for stability of constant-coefficient problems regardless of the boundary conditions.

More on Stability – Physical Concern

PDE: $u_{xx} = u_t$

Simple Explicit Method: $u_{i,j+1} = r\, u_{i-1,j} + (1 - 2r)\, u_{i,j} + r\, u_{i+1,j}$

Suppose that, at time step j: $u_{i-1,j} = 25$, $u_{i+1,j} = 25$ and $u_{i,j} = 100$

It is expected that, at the next time step, $u_{i,j+1}$ cannot be less than 25 and more than 100

The finite difference equation becomes: $u_{i,j+1} = r\,(25) + (1 - 2r)\,(100) + r\,(25)$

$$u_{i,j+1} = (1 - 2r)\,(100) + r\,(50)$$

This shows that (1 – 2 r) cannot be negative: $(1 - 2r) > 0 \quad \Rightarrow \quad r < 1/2$

This shows that $(1 – 2 r)$ cannot be negative: $(1 – 2 r) > 0$ => $r < 1/2$

For $r = 1/2$: $u_{i,j+1} = 25$ a limiting case

For $r = 1$: $u_{i,j+1} = -50$ impossible

For $r = 1/4$: $u_{i,j+1} = 50 – 50/4$ possible

For $r = 0$: $u_{i,j+1} = 100$ the other limiting case

Stability with BC's – Physical Argument

PDE: $$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \qquad 0 < x < L \;,\; t > 0$$

Difference Equation: $$T_{i\,,\,j+1} = r\,T_{i-1\,,\,j} + (1 - 2\,r)\,T_{i\,,\,j} + r\,T_{i+1\,,\,j}$$

$$\Delta x = \frac{L}{M} \quad \text{and} \quad r = \alpha \frac{\Delta t}{\Delta x^2}$$

Initial Condition: $T(x,0) = T_a \quad => \quad T_{i,0} = T_a$

Boundary Condition at x = 0:

$$-k \frac{\partial T}{\partial x} + h_c\, T(0.t) = h_c\, T_\infty \qquad \text{or}$$

$$-k \frac{\partial T}{\partial x} = h_c\, (T_\infty - T)$$

Boundary Condition at x = 0

$$-k \frac{\partial T}{\partial x} + h_c\ T(0.t) = h_c\ T_\infty \qquad \text{or}$$

$$-k \frac{\partial T}{\partial x} = h_c\ (T_\infty - T)$$

Corresponding difference form: $-k \left[\dfrac{T_{1,j} - T_{-1,j}}{2\,h} \right] + h_c\ T_{0,j} = h_c\ T_\infty$

The difference equation at i = 0 becomes:

$$T_{0\,,\,j+1} = (1 - 2\,r\,\beta_0)\,T_{0\,,\,j} + 2\,r\,T_{1\,,\,j} + 2\,r\,\gamma_0$$

where: $\beta_0 = 1 + \dfrac{\Delta x\, h_c}{k}$ and $\gamma_0 = \dfrac{\Delta x\, h_c}{k}\,T_\infty$

Suppose that, at time step j: $T_{0\,,\,j} = 100$, $T_{1\,,\,j} = 0$ and $T_{\infty} = 100$

This gives $\gamma_0 = 0$.

Then, the nodal equation (i = 0) becomes: $T_{0\,,\,j+1} = (1 - 2\,r\,\beta_0)\,(100)$

$T_{0,\,j+1}$ can only be between 0 and 100. Therefore, $(1 - 2\,r\,\beta_0) \geq 0$

$$0 < r \leq \frac{1}{2\,\beta_0} = \frac{1}{2 + 2\left(\dfrac{\Delta x\, h_c}{k}\right)}$$

This is a more restrictive condition than r < 1/2. Use the smallest r.

The Method of Lines

It applies to initial-value problems, and reduces a PDE to a system of ODE's

Example

$$u_t = u_{xx} + \cos(x)\, u_x + \left[\sin(2\,x) - \cos(t + x)\right] u$$

IC: $u(x,0) = x\,(1 - x)$

BC's: $u(0,t) = 0$

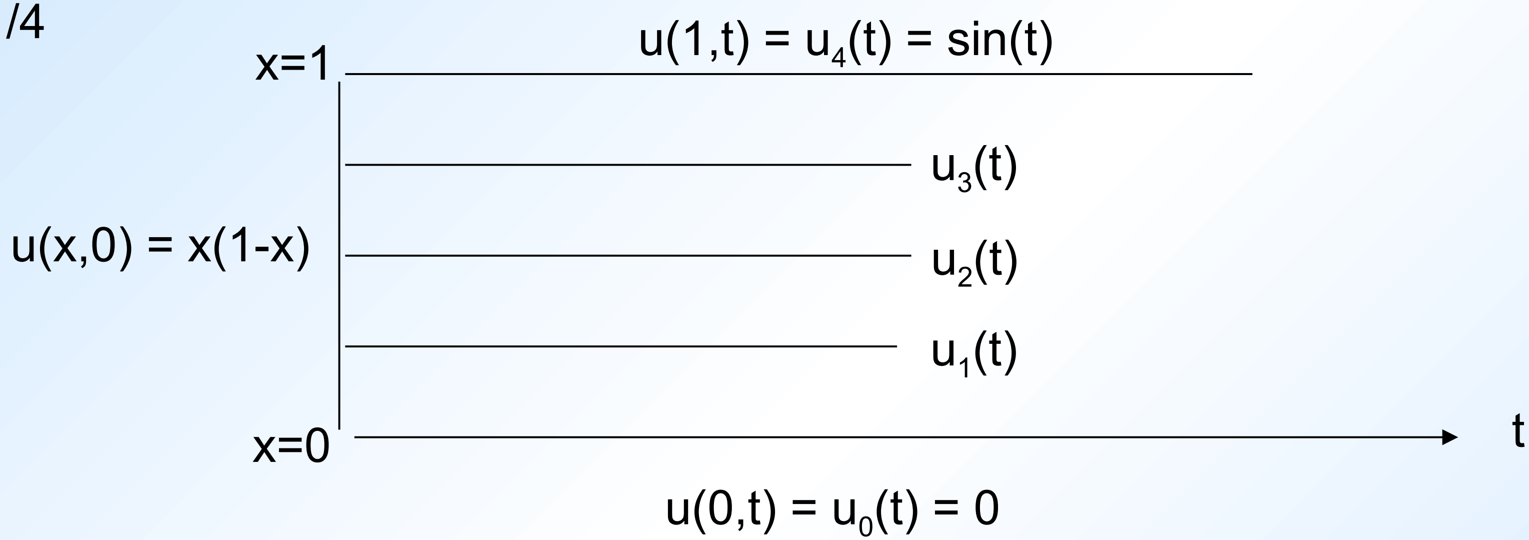
$u(1,t) = \sin(t)$

Discretisize the space variable only. Leave the time variable as it is.

Set $x = i\,h$, $i = 0,1, 2,..., n$ where $h = 1/(n+1)$

Along each line (x_i,t) for $t > 0$, we have a function $u_i(t)$.

If h = 1/4



$$\frac{du_1(t)}{dt} = \frac{u_0 - 2u_1 + u_2}{h^2} + \cos(0.25)\,\frac{u_2 - u_0}{2h} + \left[\sin(0.5) - \cos\left(t + \frac{1}{4}\right)\right] u_1$$

$$\frac{du_2(t)}{dt} = \frac{u_1 - 2u_2 + u_3}{h^2} + \cos(0.5)\,\frac{u_3 - u_1}{2h} + \left[\sin(1) - \cos\left(t + \frac{1}{2}\right)\right] u_2$$

$$\frac{du_3(t)}{dt} = \frac{u_2 - 2u_3 + u_4}{h^2} + \cos(0.75)\,\frac{u_4 - u_2}{2h} + \left[\sin(1.5) - \cos\left(t + \frac{3}{4}\right)\right] u_3$$

Initial Conditions:

$u(x,0) = x\,(1 - x)$

$u_1(0) = 0.25\,(1 - 0.25) = 0.1875$

$u_2(0) = 0.5\,(1 - 0.5) = 0.25$

$u_3(0) = 0.75\,(1 - 0.75) = 0.1875$

General Formula:

$$\frac{du_i(t)}{dt} = \frac{u_{i-1} - 2\,u_i + u_{i+1}}{h^2} + \cos\left(x_i\right)\frac{u_{i+1} - u_{i-1}}{2\,h} + \left[\sin\left(2\,x_i\right) - \cos(t + x_i)\right] u_i$$

Solve them simultaneously for all i.