PDE - INTRODUCTION

Finite difference approximations for derivatives were already in use by Euler in 1768.

$$\frac{dy}{dt} = f(t, y) \quad , \quad y(t_0) = y_0$$

$$y_{n+1} = y_n + h\, f(t_n\, , y_n)$$

For two dimensional systems, the first computational application of finite difference methods was probably carried out by Runge in 1908. He studied the numerical solution of the Poisson equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \text{constant}$$

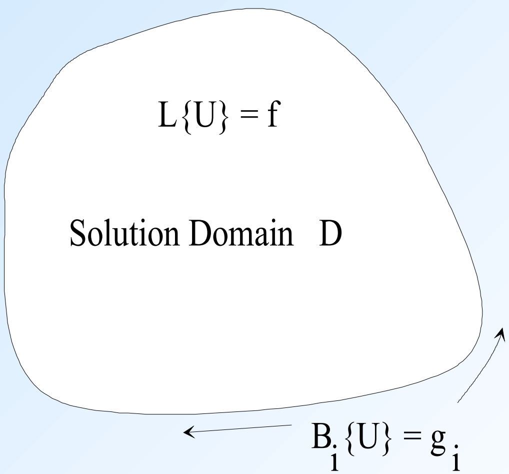
The algebraic solution of finite difference approximations is best accomplished by some *iteration* procedure. Certain classes of problems (equations) have ***natural*** numerical solutions which may be distinct from finite difference methods.

Physical Classification of PDE's:

The majority of problems in physics and engineering fall naturally into one of the three physical categories:

1. Equilibrium problems;
2. Eigenvalue problems; and
3. Propagation problems.

1. Equilibrium Problems



These are steady-state problems in which equilibrium configuration, U, in a solution domain, D, is to be determined by solving L{U(x,y)} = f within D, subject to certain Boundary Conditions (BC's) $B_i\{U\} = g_i$ on boundary of D. Very often, integration domain, D, is closed and bounded. Such problems are called boundary-value problems.

Typical examples are:	Steady viscous flow
	Steady temperature distribution
	Equilibrium stresses in elastic structures
	Steady voltage distributions

2. Eigenvalue Problems

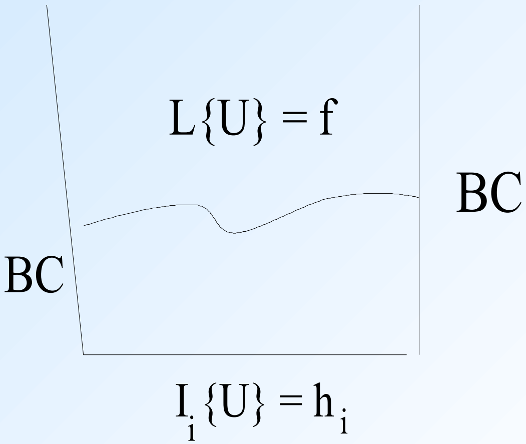
This can be tought as extension of equilibrium problems wherein critical values of certain parameters are to be determined in addition to a corresponding steady-state configuration.

$$B_i \{U\} = \lambda \, E_i \{U\}$$

Find λ and corresponding U to satisfy this equation within domain D and the BC's hold on the boundary of D.

Examples:	Buckling and stability of structures
	Resonance in acoustics and electrical circuits
	Natural frequency problems in vibrations

3. Propagation Problems



These problems are also known as initial boundary value problems

Examples:	Propagation of pressure waves in a fluid
	Propagation of heat
	Propagation of stresses and displacements
	Propagation of stresses in elastic systems and self excited vibrations

Mathematical Classification:

ODEs: An equation that relates the independent variable x, the dependent variable u and derivatives of u is called an ordinary differential equation.

Some examples of ODEs are:

$u'(x) = u$ First order, linear

$u'' + 2\,x\,u = e^{x}$ Second order, linear

$u'' + x\,(u')^{2} + \sin(x) = \ln(x)$ Second order, non-linear

$F\left(x, y, u, u', u'', ...\right) = 0$ General Form

PDE's: A partial differential equation (PDE) contains partial derivatives of the dependent variable, which is an unknown function in more than one variable x, y, ...

We will be primarily concerned with PDEs in two independent variables.

$$F\left(x, y, u, u_x, u_y\right) = 0 \qquad \text{General Form}$$

Note that one of the independent variables, x or y, can be time, t.

A solution to the PDE is a function u(x,y) which satisfies the general form for all values of the variables x and y.

Some examples of PDEs (of physical significance) are:

$u_x + u_y = 0$ Transport equation

$u_t + u\,u_x = 0$ Inviscid Burger's equation

$u_{xx} + u_{yy} = 0$ Laplace equation

$u_{tt} - u_{xx} = 0$ Wave equation

$u_t - u_{xx} = 0$ Heat equation

$u_t + u\,u_x + u_{xxx} = 0$ KdV (Korteweg–de Vries) equation

$i\,u_t - u_{xx} = 0$ Shrödinger's equation

Mathematical Classification of Second-Order PDE's

Consider the second order equation:

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x\,\partial y} + c\frac{\partial^2 u}{\partial y^2} = f$$

$$a\,u_{xx} + b\,u_{xy} + c\,u_{yy} = f\,(x, y, u, u_x, u_y)$$

where a, b, c, and f are all functions of x, y, u, u_x, and u_y, in general. The directional derivatives of u_x and u_y must also exist, i.e.,

$$d\,(u_x) = u_{xx}\,d\,x + u_{xy}\,d\,y$$

$$d\,(u_y) = u_{xy}\,d\,x + u_{yy}\,d\,y$$

In matrix notation:

$$\begin{pmatrix} a & b & c \\ d\,x & d\,y & 0 \\ 0 & d\,x & d\,y \end{pmatrix} \begin{pmatrix} u_{xx} \\ u_{xy} \\ u_{yy} \end{pmatrix} = \begin{pmatrix} f \\ d\,(u_x) \\ d\,(u_y) \end{pmatrix}$$

The solution for u_{xx}, u_{xy}, and u_{yy} exists, and it is unique unless the determinant of the coefficients matrix vanishes:

$$a\,(d\,y)^2 - b\,(d\,y)\,(d\,x) + c\,(d\,x)^2 = 0$$

This is called the characteristic equation.

If	$b^2 - 4\,a\,c > 0$	Hyperbolic PDE	Two real solutions
	$b^2 - 4\,a\,c = 0$	Parabolic PDE	Only one real solution
	$b^2 - 4\,a\,c < 0$	Elliptic PDE	No real solution

"Solution" means relation(s) between what are supposed to be unrelated (or independent) variables, x and y.

Why are we looking for a relation (or relations) between what are supposed to be independent variables, x and y? In other words, why are we trying to find a dependence between independent variables?

The objective (reason) to find such a relation is to change the PDE to an ODE when this relation is true. In other words, over what is called a characteristic curve (given by that relation), it possible to make such a change. ODE's may have exact solutions. If not, it is at least easer to numerically solve an ODE (remember R-K order 4).

That is why only the Hyperbolic type PDE's (that has two real solutions, or two relations, or two real curves, or two characteristic curves, or two characteristics between the independent variables, x and y) can be changed into ODE's.

Parabolic and Elliptic type PDE's cannot be changed into ODE's.

Examples:

Transient heat conduction equation

Propagation - Parabolic

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{\alpha} \frac{\partial U}{\partial t}$$

Steady heat conduction equation

Equilibrium - Elliptic

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$$

Compressible fluid flow or

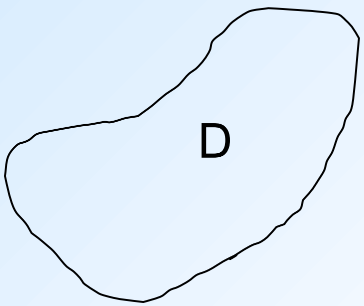
Vibration of a string

Eigenvalue - Hyperbolic

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{c_1^2} \frac{\partial^2 U}{\partial t^2}$$

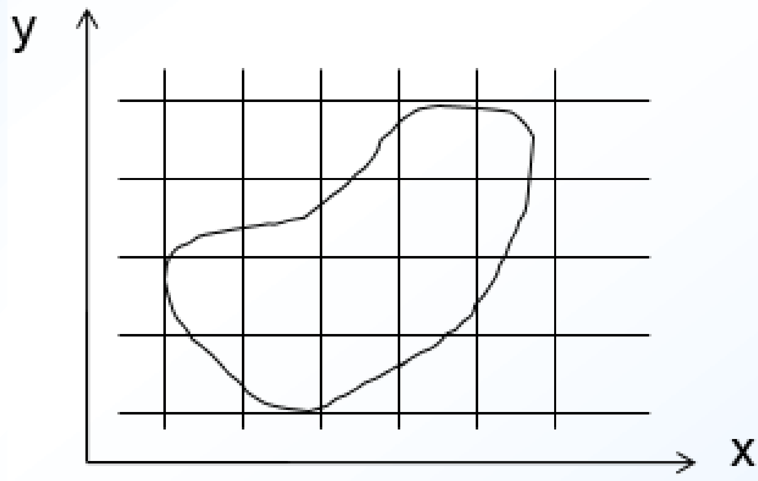
Replace all partial derivatives (in the PDE and the conditions) with equivalent finite differences.

Finite Differences:

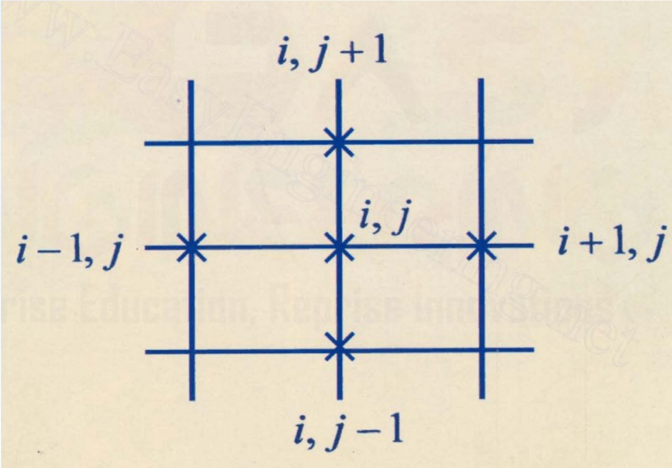


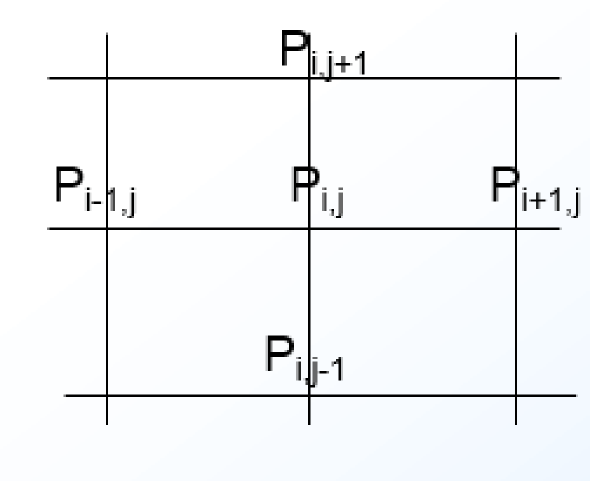
Continuous domain, D

Discrete Approximation



Grid structure, h = Δx, k = Δy





Finite Differences:

Partial derivatives can be approximated by finite differences in many ways.

Forward Difference:

$$U(x+\Delta x\,,\,y) = U(x,y) + \Delta x\,\frac{\partial U}{\partial x} + \frac{\Delta x^2}{2!}\,\frac{\partial^2 U}{\partial x^2} + \frac{\Delta x^3}{3!}\,\frac{\partial^3 U}{\partial x^3} + O(\Delta x^4)$$

$$\frac{\partial U}{\partial x} = \frac{U(x+\Delta x\,,\,y) - U(x,y)}{\Delta x} + O(\Delta x)$$

$$\left.\frac{\partial U}{\partial x}\right|_{i,j} = \frac{U_{i+1,j} - U_{i,j}}{h} + O(h)$$

Backward Difference:

$$\left.\frac{\partial U}{\partial x}\right|_{i,j} = \frac{U_{i,j} - U_{i-1,j}}{h} + O(h)$$

Central Differences:

$$\left.\frac{\partial U}{\partial x}\right|_{i,j} = \frac{U_{i+1,j} - U_{i-1,j}}{2h} + O(h^2)$$

$$\left.\frac{\partial U}{\partial y}\right|_{i,j} = \frac{U_{i,j+1} - U_{i,j-1}}{2k} + O(k^2)$$

$$\left.\frac{\partial^2 U}{\partial x^2}\right|_{i,j} = \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2} + O(h^2)$$

$$\left.\frac{\partial^2 U}{\partial y^2}\right|_{i,j} = \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{k^2} + O(k^2)$$

If h = k:

$$\nabla^2 U = \left.\frac{\partial^2 U}{\partial x^2}\right|_{i,j} + \left.\frac{\partial^2 U}{\partial y^2}\right|_{i,j} = \frac{U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} - 4\,U_{i,j}}{h^2} + O(h^2)$$

$$\left.\frac{\partial^2 U}{\partial x\,\partial y}\right|_{i,j} = \frac{U_{i-1,j+1} + U_{i+1,j+1} + U_{i-1,j-1} + U_{i+1,j-1}}{4\,h^2} + O(h^2)$$

Note that these are the common approximations, but not the only ones.

First Derivatives

$f'(x_0) \cong \frac{1}{h}\left(f_1 - f_0\right)$	$E'(x_0) = \frac{-h}{2} f''(\xi)$	One-term forward
$f'(x_0) \cong \frac{1}{2h}\left(f_1 - f_{-1}\right)$	$E'(x_0) = \frac{-h^2}{6} f'''(\xi)$	One-term central
$f'(x_0) \cong \frac{1}{2h}\left(-f_2 + 4f_1 - 3f_0\right)$	$E'(x_0) = \frac{h^2}{3} f'''(\xi)$	Two-term forward
$f'(x_0) \cong \frac{1}{12h}\left(-f_2 + 8f_1 - 8f_{-1} + f_{-2}\right)$	$E'(x_0) = \frac{-h^4}{30} f^{(5)}(\xi)$	Two-term central
$f'(x_0) \cong \frac{1}{6h}\left(2f_3 - 9f_2 + 18f_1 - 11f_0\right)$	$O(h^3)$	Three-term forward

Note that, using central differences, we get better accuracy with two-terms than we do with four terms using forward differences.

Second Derivatives

$$f''(x_0) \cong \frac{1}{h^2}\left(f_2 - 2\,f_1 + f_0\right) \qquad E''(x_0) = h\,f'''(\xi)$$

Two-term forward

$$f''(x_0) \cong \frac{1}{h^2}\left(f_1 - 2f_0 + f_{-1}\right) \qquad E''(x_0) = \frac{-h^2}{12}\,f^{(4)}(\xi)$$

Two-term central

$$f''(x_0) \cong \frac{1}{h^2}\left(-f_3 + 4\,f_2 - 5\,f_1 + 2\,f_0\right) \qquad O(h^2)$$

Three-term forward

$$f''(x_0) \cong \frac{1}{12\,h^2}\left(-f_2 + 16\,f_1 - 30\,f_0 + 16\,f_{-1} - f_{*2}\right) \qquad O(h^4)$$

Three-term central

Numerical Solution Methods of PDE's:

- Finite Differences – simplest to learn and use
- Finite Elements
- Finite Volumes

} widely used in engineering and in computational_fluid_dynamics

- Gradient discretization
- Spectral methods – often use fast Fourier transforms
- Method of lines
- Mesh free methods
- Domain decomposition
- Multigrid
- Others