# CLASSIFICATION OF DIFFERENTIAL EQUATIONS

| | | |
|---|---|---|
| Differential Equations | Ordinary | ODE |
| | Partial | PDE |

| | | |
|---|---|---|
| ODE | Linear (in y and its derivatives) | $y' + A x^2 y = f(x)$ |
| | Non-linear (in y and its derivatives) | $(y')^2 + A y y' = f(x)$ |

| | | |
|---|---|---|
| ODE | Homogeneous | $\ldots\ldots = 0$ |
| | Non-homogeneous | $\ldots\ldots = f(x) \neq 0$ |

| | |
|---|---|
| System of ODE's | $y' = f(x, y, z)$<br>$z' = g(x, y, z)$ |

| | | |
|---|---|---|
| ODE | Initial-value Problem | depending on the conditions |
| | Boundary-value Problem | |

PDE's:

| | | |
|---|---|---|
| Physical Problems | Propagation Problem | open domain |
| | Equilibrium Problem | closed domain |
| | Eigenvalue Problem | |

| | |
|---|---|
| Solution Methods | Marching Numerical Method |
| | Equilibrium Numerical Method |

PDE's:

| | | |
|---|---|---|
| Mathematical Concern | Parabolic | Often a propagation problem |
| | Elliptic | Often an equilibium problem |
| | Hyperbolic | Often an eigenvalue problem |

# LIPSHITZ CONDITION

Theorem: on existence of a solution for first-order ODE:

$$\frac{dy(t)}{dt} = f\,(t,y)$$

If $f\,(t,y)$ is continuous on $a \le t \le b$, and there exists a constant $L$ such that

$$\left| f\,(t,y) - f\,(t,z) \right| \le L \left| y - z \right|$$

for all $t \,\varepsilon\, [a\,,\,b]$, and all real $y$ and $z$, then the initial-value problem

$$\frac{dy(t)}{dt} = f\,(t,y) \qquad \text{with} \qquad y\,(t_0) = y_0$$

where $t_0$ in $[a\,,\,b]$ possesses a unique solution. This is called Lipschitz condition.

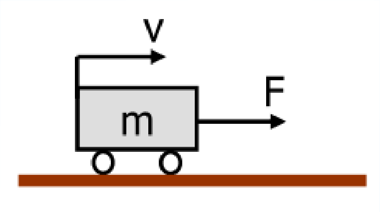
Rudolph Otto Sigismund Lipschitz

German mathematician

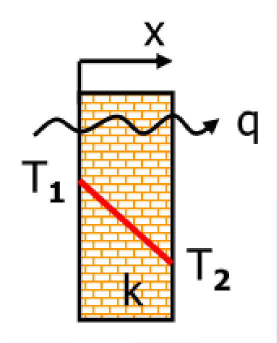
1832 – 1903

## Examples of ODE's

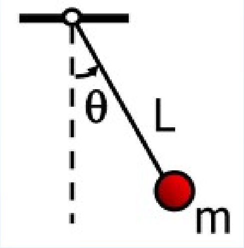
Newton's 2nd law of motion $F = m \dfrac{dv}{dt}$




Fourier's law of heat conduction $q = -k \dfrac{dT}{dx}$



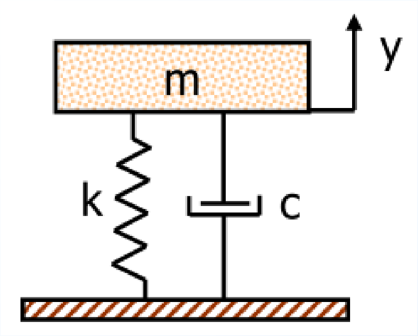

Swinging pendulum $\dfrac{d^2\theta}{dt^2} + \dfrac{g}{L}\sin(\theta) = 0$



## Examples of ODE's

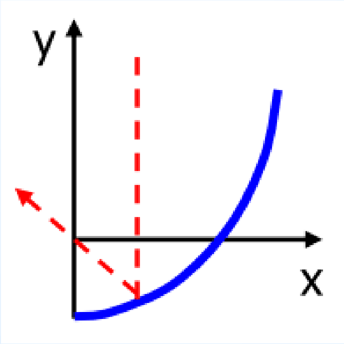
Mass-spring-damper system

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + k\,y = 0$$




Collector of a solar heater

$$x\left(\frac{dy}{dx}\right)^2 - 2\,y\,\frac{dy}{dx} = 0$$



## First–order, Ordinary Differential Equation

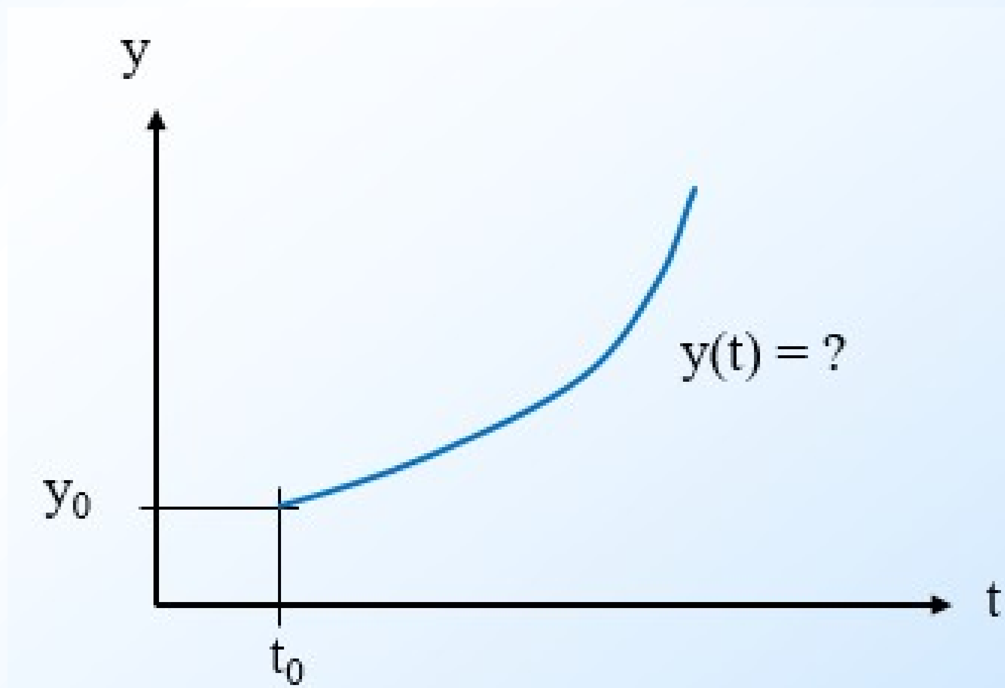
$$\frac{dy(t)}{dt} = f(t,y) \quad , \quad y(t_0) = y_0$$

$$\int_{y_0}^{y} dy(t) = \int_{t_0}^{t} f(t,y)\,dt$$

$$y(t) - y_0 = \int_{t_0}^{y} f(t,y)\,dt$$

$$y(t) = y_0 + \int_{t_0}^{y} f\big(t,y(t)\big)\,dt$$

How to solve?



## TAYLOR SERIES

$$\frac{dy(t)}{dt} = f(t,y) \quad , \quad y(t_0) = y_0$$

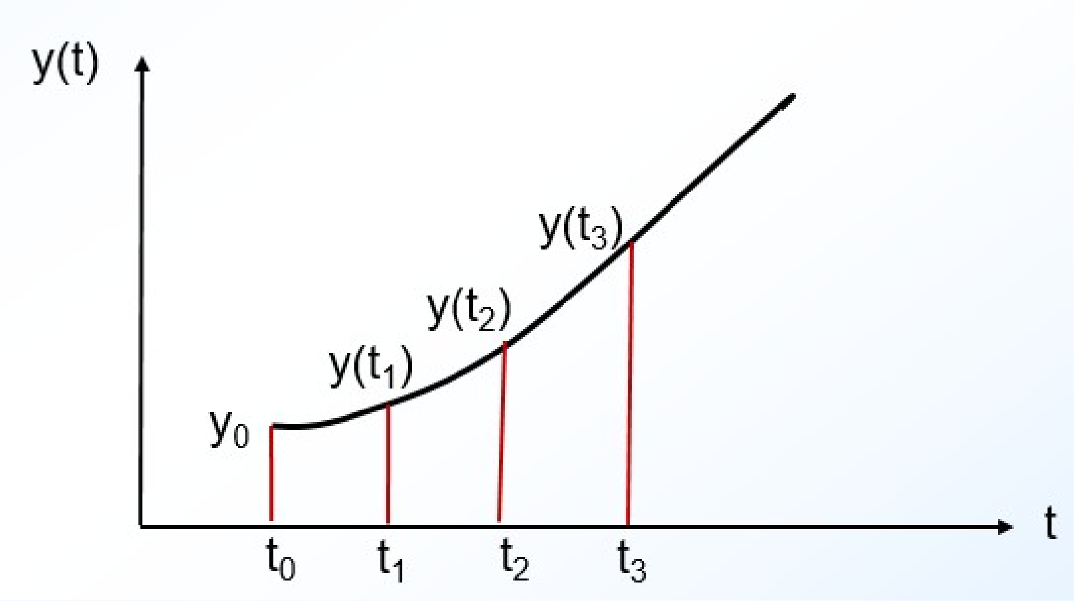
$$y(t) = y(t_0) + (t - t_0)\, f(t_0,y_0) + \frac{(t - t_0)^2}{2!}\, y''(t_0) + \ldots + \frac{(t - t)^{n+1}}{(n + 1)!}\, y^{(n+1)}(\xi)$$

$$y(t_1) = y(t_0) + (t_1 - t_0)\, f(t_0,y_0) + \frac{(t_1 - t_0)^2}{2!}\, y''(t_0) + \ldots + \frac{(t_1 - t_0)^{n+1}}{(n + 1)!}\, y''(\xi)$$

$$y(t_2) = y(t_0) + (t_2 - t_0)\, f(t_0,y_0) + \frac{(t_2 - t_0)^2}{2!}\, y''(t_0) + \ldots + \frac{(t_2 - t_0)^{n+1}}{(n + 1)!}\, y^{n+1}(\xi)$$

$$y(t_n) = y(t_0) + (t_n - t_0)\, f(t_0,y_0) + \frac{(t_n - t_0)^2}{2!}\, y''(t_0) + \ldots + \frac{(t_n - t_0)^{n+1}}{(n + 1)!}\, y^{n+1}(\xi)$$

All the y's can be found. But ...




$y(t_0) = y_0$ IC

$y(t_1) = y_1 = ?$

$y(t_2) = y_2 = ?$

.......

$y(t_n) = y_n = ?$

Too much error because $t_0$ and $t_n$ are too far apart

## RUNNING TAYLOR SERIES

$$\frac{dy(t)}{dt} = f(t,y) \quad , \quad y(t_0) = y_0$$

$$y(t) = y(t_0) + (t - t_0)\, f(t_0,y_0) + \frac{(t - t_0)^2}{2!}\, y''(t_0) + \dots + \frac{(t - t)^{n+1}}{(n+1)!}\, y^{(n+1)}(\xi)$$

$$y(t_1) = y(t_0) + (t_1 - t_0)\, f(t_0,y_0) + \frac{(t_1 - t_0)^2}{2!}\, y''(t_0) + \dots + \frac{(t_1 - t_0)^{n+1}}{(n+1)!}\, y^{n+1}(\xi)$$

$$y(t_2) = y(t_1) + (t_2 - t_1)\, f(t_1,y_1) + \frac{(t_2 - t_1)^2}{2!}\, y''(t_1) + \dots + \frac{(t_2 - t_1)^{n+1}}{(n+1)!}\, y^{n+1}(\xi)$$

$$y(t_n) = y(t_{n-1}) + (t_n - t_{n-1})\, f(t_{n-1},y_{n-1}) + \frac{(t_n - t_{n-1})^2}{2!}\, y''(t_{n-1}) + \dots + \frac{(t_n - t_{n-1})^{n+1}}{(n+1)!}\, y^{n+1}(\xi)$$

## RUNNING TAYLOR SERIES

$$\frac{dy(t)}{dt} = f(t,y) \quad , \quad y(t_0) = y_0$$

$$y(t) = y(t_0) + (t - t_0)\, f(t_0,y_0) + \frac{(t - t_0)^2}{2!}\, y''(t_0) + \ldots + \frac{(t - t)^{n+1}}{(n+1)!}\, y^{(n+1)}(\xi)$$

$$y(t_{n+1}) = y(t_n) + (t_{n+1} - t_n)\, f(t_n,y_n) + \frac{(t_{n+1} - t_n)^2}{2!}\, y''(t_n) + \ldots + \frac{(t_{n+1} - t_n)^{n+1}}{(n+1)!}\, y^{(n+1)}(\xi)$$

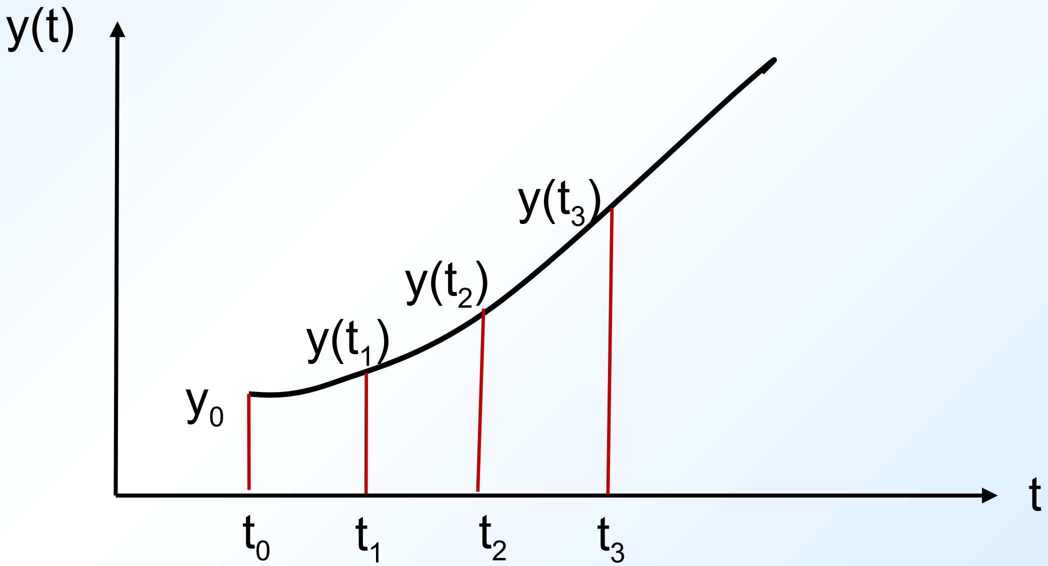
$$y_{n+1} = y_n + h\, f(t_n,y_n) + \frac{h^2}{2!}\, y''(t_n) + \ldots + \frac{h^{n+1}}{(n+1)!}\, y^{(n+1)}(\xi)$$

$$y_{n+1} = y_n + h\, f(t_n,y_n) + \frac{h^2}{2!} \left[ \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}\, f \right]\Bigg|_{t_n\, ,\, y_n} + \ldots + \frac{h^{n+1}}{(n+1)!}\, y^{(n+1)}(\xi)$$

Given: $\dfrac{dy(t)}{dt} = f\big(t, y(t)\big) \quad , \quad y(t_0) = y_0$

Question: $y(t) = ?$

Choose $h = \Delta t$ $\quad y(t_1) = ? \quad , \quad y(t_2) = ? \quad , \; \ldots \; , \quad y(t_n) = ?$




$y(t_0) = y_0$ IC

$y(t_1) \cong y_1 = ?$

$y(t_2) \cong y_2 = ?$

...

$y(t_n) \cong y_n = ?$

Given: $$\frac{dy(t)}{dt} = f\big(t,y(t)\big) \quad , \quad y(t_0) = y_0$$

$$\int_{y_0}^{y} dy(t) = \int_{t_0}^{t} f\big(t,y(t)\big)\, dt$$

$$y(t) - y_0 = \int_{t_0}^{t} f\big(t,y(t)\big)\, dt \quad => \quad y(t) = y_0 + \underbrace{\int_{t_0}^{t} f\big(t,y(t)\big)\, dt}_{?}$$

$$y(t_{n+1}) - y(t_n) = \underbrace{\int_{t_n}^{t_{n+1}} f\big(t,y(t)\big)\, dt}_{?} \quad => \quad y_{n+1} = y_n + \int_{t_n}^{t_{n+1}} f\big(t,y(t)\big)\, dt$$

$$y_{n+1} = y_n + \int_{t_n}^{t_{n+1}} f\big(t,y(t)\big)\, dt \quad => \quad y_{n+1} = y_n + f\big(t_n,y_n\big) \underbrace{\int_{t_n}^{t_{n+1}} dt}_{\Delta t = h}$$

Leonhard Euler

Swiss mathematician

1707 - 1783

**EULER'S METHOD:** $y_{n+1} = y_n + h\, f(t_n, y_n)$

Local Error: $$E_{local} = \frac{h^2}{2!}\, y''(\xi) \quad , \qquad t_{n+1} < \xi < t_n$$

Total or Overall Error:
$$E_{total} = \sum \frac{h^2}{2!}\, y''(\xi_n)$$
$$= \sum \frac{h}{2!}\, y''(\xi_n)\, h \;\le\; \frac{h}{2}\, \text{Max}\,(y'') \sum h$$
$$\le \frac{h}{2}\, \text{Max}\,(y'')\,(t_n - t_0)$$
$$\propto h^1$$

Euler's method is an order 1 method, O(1) or O($h^1$) , i.e., the overall error is proportional to the first power of the step size, h.

## Interpretation of the Euler's method

Given $\dfrac{dy(t)}{dt} = f\big(t,y(t)\big) \quad , \quad y(t_0) = y_0$ Find y(t)

Or, define $h = \Delta t = t_1 - t_0 = t_2 - t_1 = \ldots,$ and find $y_1 = y(t_1),\ y_2 = y(t_2),\ \ldots$

Use only the first to terms of the running Taylor series:

f(t,y) is assumed to remain constant at the beginning of the interval

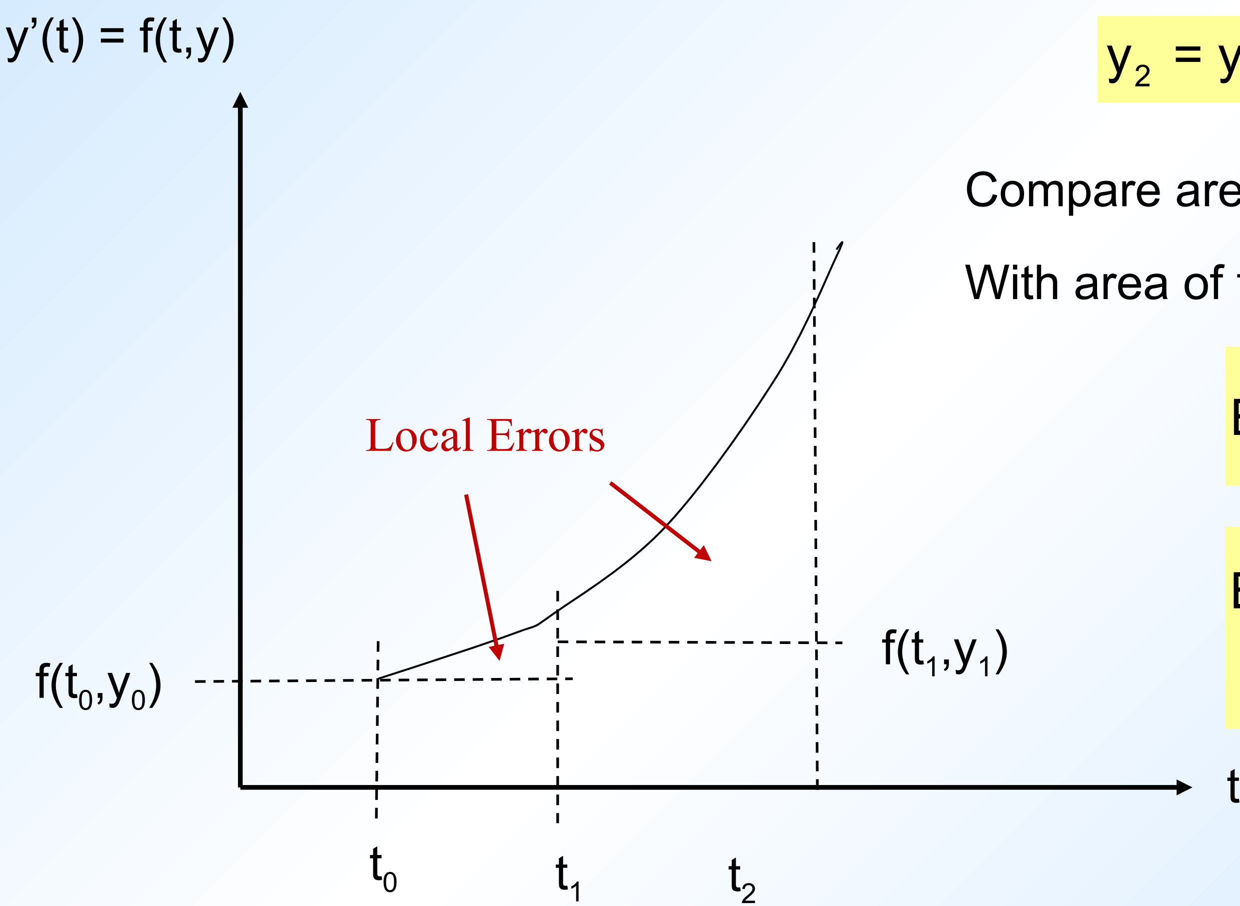
Why at the beginning?

$$y_1 = y_0 + h\,f\big(t_0,y_0\big)$$

$$\int_{y_0}^{y_1} dy(t) = \int_{t_0}^{t_1} f\big(t,y(t)\big)dt$$

$$y_1 - y_0 \cong f\big(t_0,y_0\big)\int_{t_0}^{t_1} dt = h\,f\big(t_0,y_0\big)$$

$$y_1 \cong y_0 + h\,f\big(t_0,y_0\big)$$




$$y_1 = y_0 + h\,f\left(t_0,y_0\right)$$

$$y_2 = y_1 + h\,f\left(t_1,y_1\right)$$

Compare area under the curve (exact)

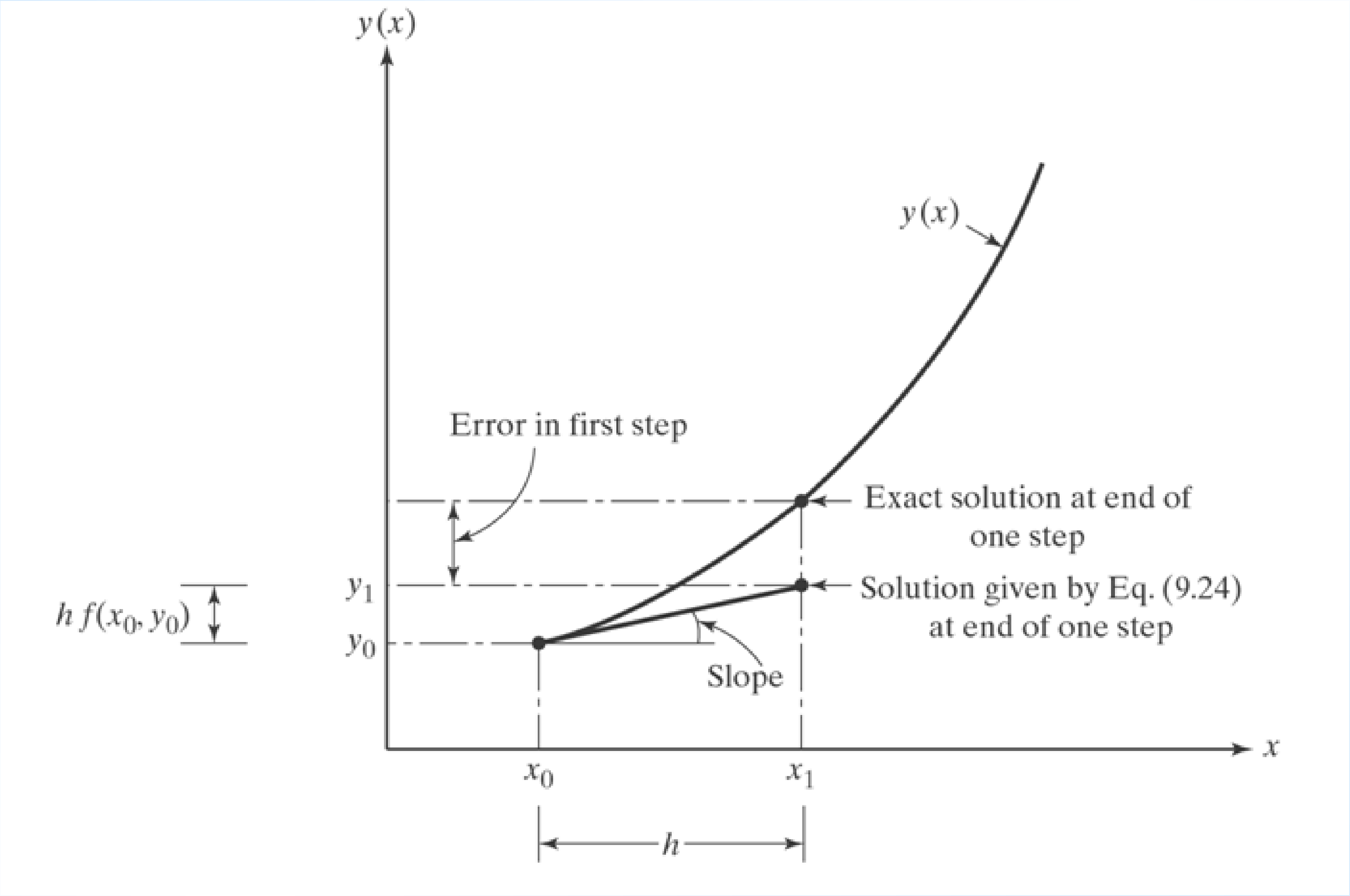
With area of the rectangle (approximate)

$$E_{total} = \sum \frac{h^2}{2!}\,y''(\xi_n)$$

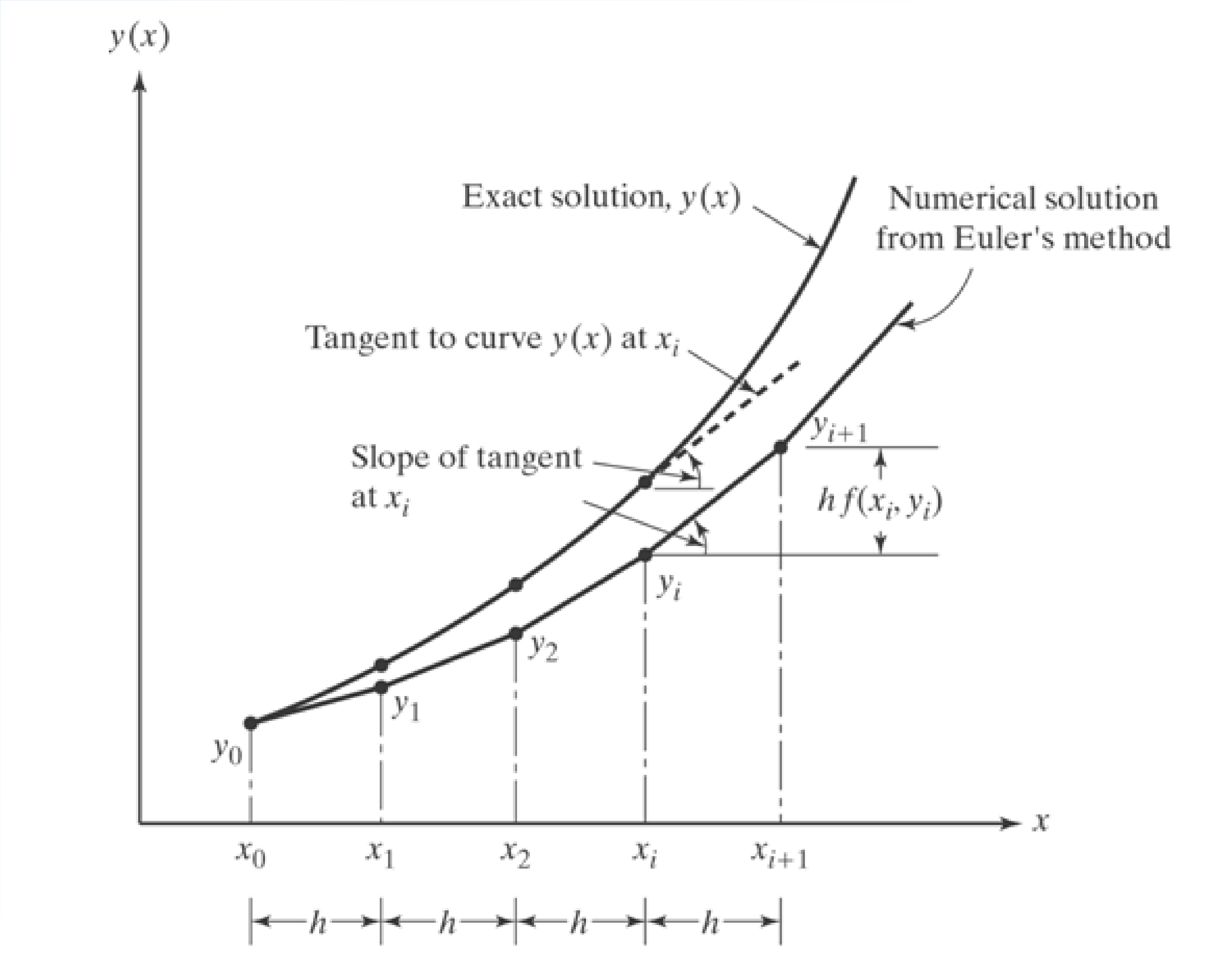
$$E_{total} \le \frac{h}{2}\,\text{Max}\,(y'')\,(t_n - t_0)$$

$$\propto h^1$$

O(1) method




Geometrical Interpretation of Euler's method




Accumulation of Error in Euler's method

# EXAMPLE ON EULER'S METHOD

$$\frac{d\,y(t)}{d\,t} = -a\,y(t) \quad , \quad y(0) = 1$$

$$y_{exact} = e^{-a\,t}$$

Euler's Method: $y_{n+1} = y_n + h\,f(t_n, y_n)$

| n | 0 | 1 | 2 | 3 | 4 | 5 | 10 | 100 | 1000 |
|---|---|---|---|---|---|---|---|---|---|
| t / a | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 1.0 | 10.0 | 100.0 |
| $y_{exact}$ | 1 | 0.905 | 0.819 | 0.741 | 0.670 | 0.607 | 0.368 | $4.5\ 10^{-5}$ | $3.7\ 10^{-44}$ |
| $y_{Euler}$ | 1 | 0.900 | 0.810 | 0.729 | 0.656 | 0.590 | 0.349 | $2.6\ 10^{-5}$ | $1.0\ 10^{-46}$ |

# EXAMPLE ON MODIFICATIONS OF EULER'S METHOD

$$y_{n+1} = y_n + h\, f\left(t_n + \frac{h}{2} \,,\, \frac{y_n + y_{n+1}}{2}\right)$$

$$\frac{d\,y(t)}{d\,t} = -\,a\,y(t) \quad , \quad y(0) = 1$$

$$y_{exact} = e^{-a\,t}$$

Euler's Method: $y_{n+1} = y_n + h\, f(t_n, y_n)$

Mod. 1 $y_{n+1} = y_n + h\, f(t_{n+1}, y_{n+1})$

Mod. 2 $y_{n+1} = y_n + h\, f\left(t_n + \frac{h}{2} \,,\, \frac{y_n + y_{n+1}}{2}\right)$

# EXAMPLE ON MODIFICATIONS OF EULER'S METHOD

| n | 0 | 1 | 2 | 3 | 4 | 5 | 10 | 100 | 1000 |
|---|---|---|---|---|---|---|---|---|---|
| t / a | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 1.0 | 10.0 | 100.0 |
| $y_{exact}$ | 1 | 0.905 | 0.819 | 0.741 | 0.670 | 0.607 | 0.368 | $4.5\ 10^{-5}$ | $3.7\ 10^{-44}$ |
| $y_{Euler}$ | 1 | 0.900 | 0.810 | 0.729 | 0.656 | 0.590 | 0.349 | $2.6\ 10^{-5}$ | $1.0\ 10^{-46}$ |
| $y_{end}$ | 1 | 0.909 | 0.826 | 0.751 | 0.683 | 0.621 | 0.386 | $7.3\ 10^{-5}$ | $4.1\ 10^{-42}$ |
| $y_{middle}$ | 1 | 0.905 | 0.819 | 0.741 | 0.670 | 0.606 | 0.368 | $4.6\ 10^{-5}$ | $3.4\ 10^{-44}$ |

# Runge-Kutta Methods

Taylor Series Order 2:

$$y_{n+1} = y_n + h\, f(t_n,y_n) + \frac{h^2}{2!} y''(t_n) + R$$

$$y_{n+1} = y_n + h\, f(t_n,y_n) + \frac{h^2}{2!} \boxed{\left[ \left.\frac{\partial f}{\partial t}\right|_{t_n,y_n} + \left.\frac{\partial f}{\partial y}\right|_{t_n,y_n} f(t_n,y_n) \right]} + R$$

Runge-Kutta Order 2:

$$y_{n+1} = y_n + a\, h\, f(t_n,y_n) + b\, h\, f(t^*,y^*) + R$$

$$t^* = t_n + \alpha\, h \qquad y^* = y_n + \beta\, h\, f(t_n,y_n)$$

Find the fractions, a, b, α, and β such that **R's are the same**

**Carl David Tolmé Runge**

German Scientist

1856 – 1927

**Martin Wilhelm Kutta**

German Mathematician

1867 – 1944)

## Runge-Kutta Methods

$$y_{n+1} = y_n + a\,h\,f(t_n,y_n) + b\,h\,f\Big[\underbrace{t_n+\alpha\,h}_{t^*}\,,\,\underbrace{y_n+\beta\,h\,f(t_n,y_n)}_{y^*}\Big] + R$$

Expand f (t*,y*) in Taylor Series around $(t_n,y_n)$ and substitute

$$f\left[t_n+\alpha\,h\,,\,y_n+\beta\,h\,f(t_n,y_n)\right] = f(t_n,y_n) + \alpha\,h\,\frac{\partial f}{\partial t} + \beta\,h\,f(t_n,y_n)\,\frac{\partial f}{\partial y} + \ldots$$

$$y_{n+1} = y_n + a\,h\,f(t_n,y_n) + b\,h\left[f(t_n,y_n) + \alpha\,h\,\frac{\partial f}{\partial t} + \beta\,h\,f(t_n,y_n)\,\frac{\partial f}{\partial y} + \ldots\right] + R$$

$$y_{n+1} = y_n + h\,f(t_n,y_n) + \frac{h^2}{2!}\left[\left.\frac{\partial f}{\partial t}\right|_{t_n,y_n} + \left.\frac{\partial f}{\partial y}\right|_{t_n,y_n} f(t_n,y_n)\right] + R$$

$$y_{n+1} = y_n + h\, f(t_n,y_n) + \frac{h^2}{2!}\left[\left.\frac{\partial f}{\partial t}\right|_{t_n,y_n} + \left.\frac{\partial f}{\partial y}\right|_{t_n,y_n} f(t_n,y_n)\right] + R$$

$$y_{n+1} = y_n + a\, h\, f(t_n,y_n) + b\, h\left[f(t_n,y_n) + \alpha\, h \frac{\partial f}{\partial t} + \beta\, h\, f(t_n,y_n)\frac{\partial f}{\partial y} + \cancel{\ldots}\right] + R$$

Equate the coefficients of the same powers of h:

$$\left.\begin{array}{ll} \text{For } h^1: & 1 = a + b \\ \text{For } h^2: & \tfrac{1}{2} = b\,\alpha \quad \text{and} \quad \tfrac{1}{2} = b\,\beta \end{array}\right\}$$

Choose any one of the unknowns arbitrarily, That sets the other three

## 2nd Order RUNGE-KUTTA Methods

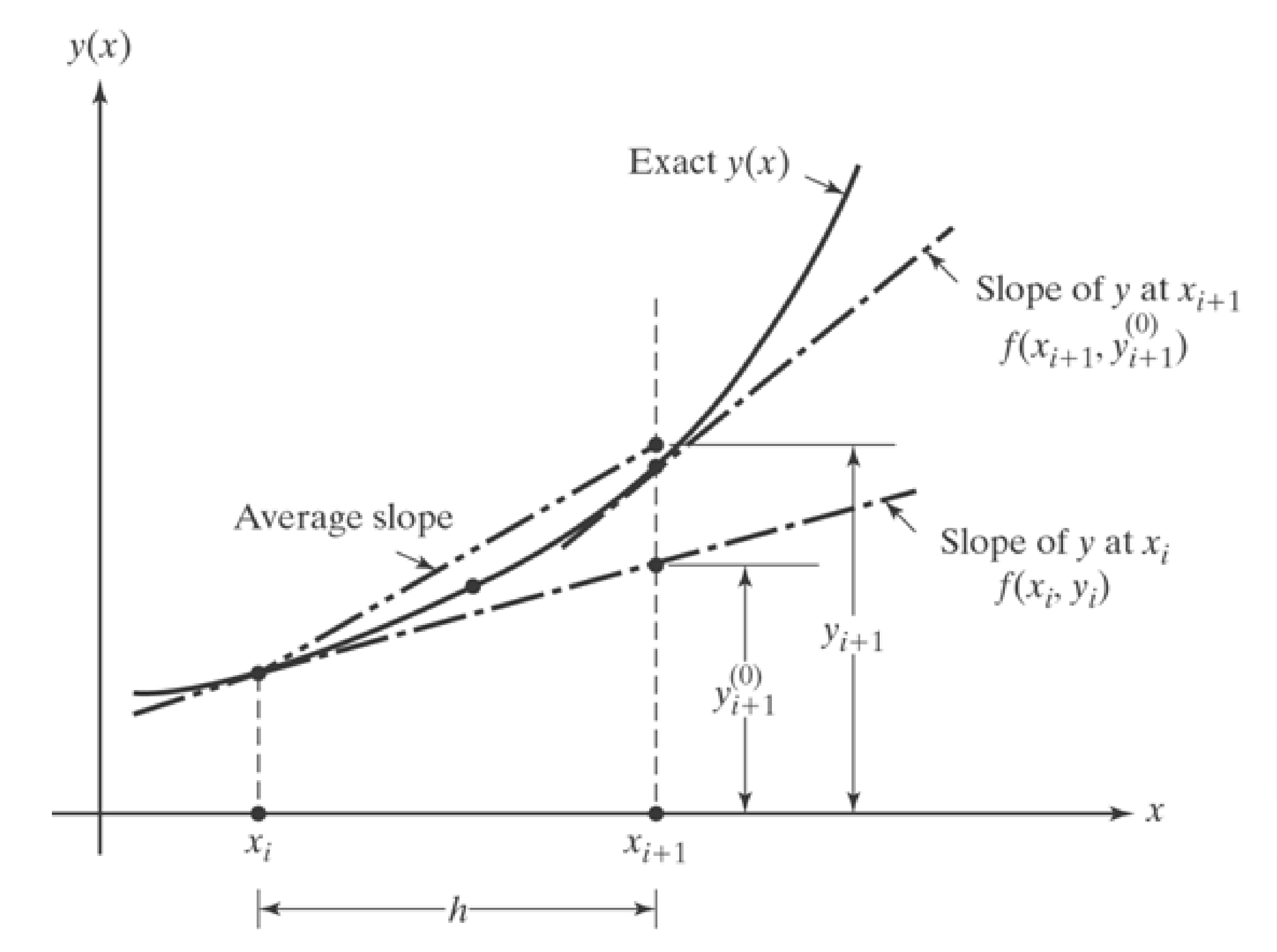
Euler's Method: $y_{n+1} = y_n + h\, f\,(t_n\,,\, y_n)$ 1st Order

Heun's Method $$y_{n+1} = y_n + \frac{h}{2}\left[f(t_n\,,\, y_n) + f\left(t_{n+1}\,,\, y_n + h\, f\,(t_n\,,\, y_n)\right)\right]$$

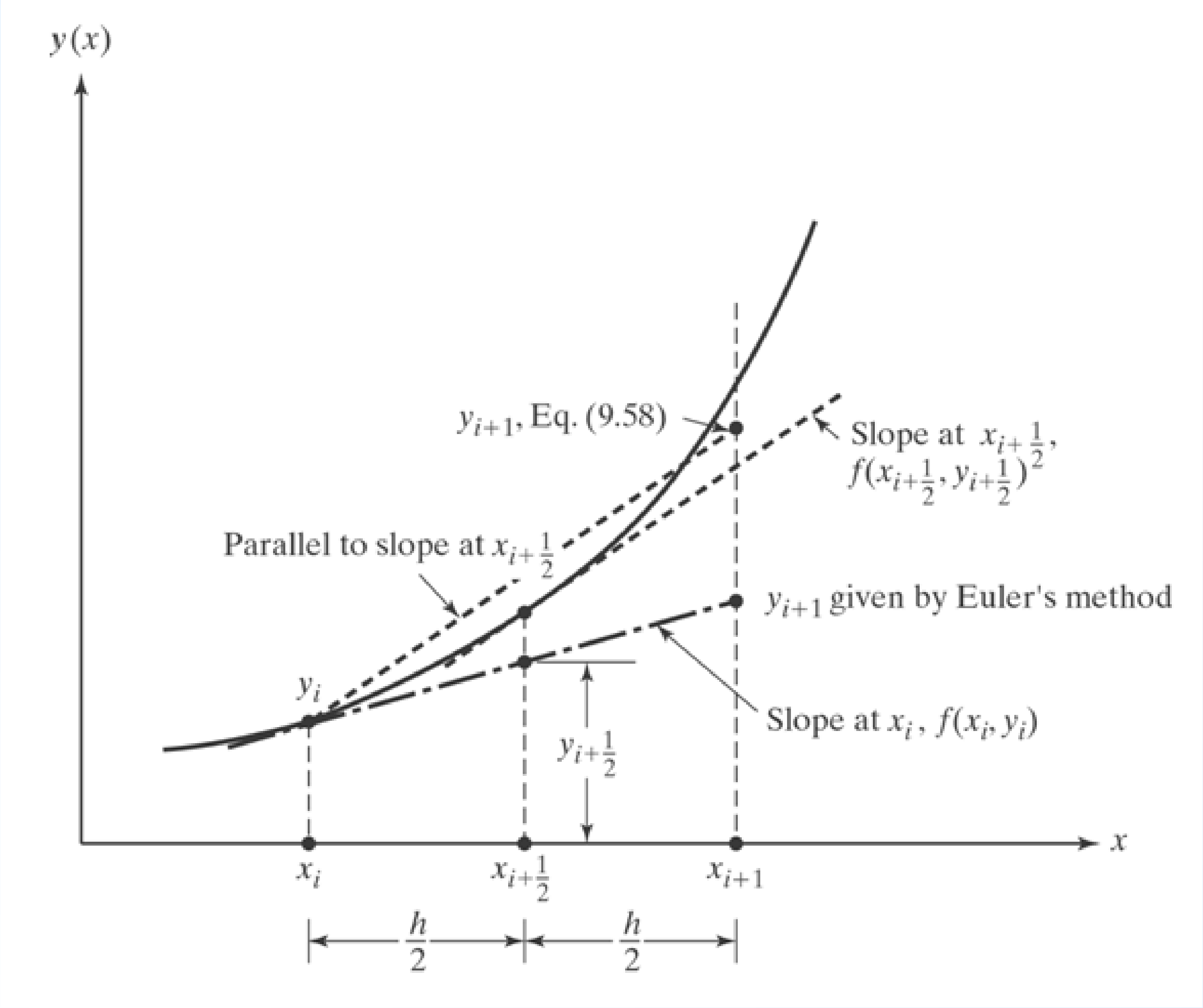
The Modified Euler's Method $$y_{n+1} = y_n + h\left[f\,(t_n + \frac{h}{2}\,,\, y_n + \frac{h}{2}\, f\,(t_n, y_n)\right]$$

**Karl Heun**

German Mathematician
1859 - 1929




## Geometrical Interpretation of Heun's method




## Geometrical Interpretation of the Modified Euler's method

## Running Taylor Series:

$$y_{n+1} = y_n + h\, y'(t_n) + \frac{h^2}{2!}\, y''(t_n) + \frac{h^3}{3!}\, y'''(t_n) + \frac{h^4}{4!}\, y^{(4)}(t_n) + \frac{h^5}{5!}\, y^{(5)}(\xi)$$

## Runge-Kutta order 4:

Local Error

$$y_{n+1} = y_n + a\, h\, f(t_n, y_n) + b\, h\, f(t_n^{*}, y_n^{*}) + c\, h\, f(t_n^{**}, y_n^{**}) + c\, h\, f(t_n^{***}, y_n^{***}) + \frac{h^5}{5!}\, y^{(5)}(\xi)$$

$$t_n^{*} = t_n + \alpha\, h \qquad y_n^{*} = y_n + \beta\, h\, f(t_n, y_n)$$

$$t_n^{**} = t_n + \gamma\, h \qquad y_n^{**} = y_n + \eta\, h\, f(t_n, y_n)$$

$$t_n^{***} = t_n + \nu\, h \qquad y_n^{**} = y_n + \varepsilon\, h\, f(t_n, y_n)$$

10 unknowns

Equate the coefficients

of the same powers of h

9 relations

## Fourth-Order Runge-Kutta Methods

**Method Attributed to Runge**:

$$y_{n+1} = y_n + \frac{h}{6}\left(K_1 + 2K_2 + 2K_3 + K_4\right)$$

$$K_1 = f(t_n , y_n)$$

$$K_2 = f\left(t_n + \frac{h}{2} , y_n + \frac{h}{2}K_1\right)$$

$$K_3 = f\left(t_n + \frac{h}{2} , y_n + \frac{h}{2}K_2\right)$$

$$K_4 = f(t_n + h , y_n + hK_3)$$

This method reduces to Simpson's one-third rule when $f(t,y) = f(t)$.

## Method Attributed to Kutta:

$$y_{n+1} = y_n + \frac{h}{8}\left(K_1 + 3\,K_2 + 3\,K_3 + K_4\right)$$

$$K_1 = f\,(t_n\,,\,y_n)$$

$$K_2 = f\left(t_n + \frac{h}{3}\,,\; y_n + \frac{h}{3}K_1\right)$$

$$K_3 = f\left(t_n + \frac{2\,h}{3}\,,\; y_n - \frac{h}{3}K_1 + h\,K_2\right)$$

$$K_4 = f\,(t_n + h\,,\; y_n + h\,K_1 - h\,K_2 + h\,K_3)$$

This method reduces to Simpson's 3/8 rule when $f\,(t,y) = f\,(t)$.

## Runge-Kutta-Gill Method:

$$y_{n+1} = y_n + \frac{h}{6}\left(K_1 + 2\left(1 - \frac{1}{\sqrt{2}}\right)K_2 + 2\left(1 + \frac{1}{\sqrt{2}}\right)K_3 + K_4\right)$$

$$K_1 = f(t_n , y_n)$$

$$K_2 = f\left(t_n + \frac{h}{2} , y_n + \frac{h}{2}K_1\right)$$

$$K_3 = f\left(t_n + \frac{h}{2} , y_n + \left(-\frac{1}{2} + \frac{1}{\sqrt{2}}\right)h\,K_1 + \left(1 - \frac{1}{\sqrt{2}}\right)h\,K_2\right)$$

$$K_4 = f\left(t_n + h , y_n - \frac{1}{\sqrt{2}}h\,K_2 + \left(1 + \frac{1}{\sqrt{2}}\right)h\,K_3\right)$$

This is one of the most widely used fourth-order methods. The constants are selected to reduce the amount of storage required in the solution of a large number of simultaneous first-order differential equations.

# EXAMPLE ON RUNGE-KUTTA METHODS

Euler's Method: $y_{n+1} = y_n + h\, f(t_n\,,\, y_n)$

Heun's Method $$y_{n+1} = y_n + \frac{h}{2}\left[f(t_n\,, y_n) + f\big(t_{n+1}\,, y_n + f(t_n\,, y_n)\big)\right]$$

The Runge-Kutta Order 4 $$y_{n+1} = y_n + \frac{h}{6}(K_1 + 2\,K_2 + 2\,K_3 + K_4)$$

$$K_1 = f(t_n\,,\, y_n)$$

$$K_2 = f\left(t_n + \frac{h}{2}\,,\; y_n + \frac{h}{2}K_1\right)$$

$$K_3 = f\left(t_n + \frac{h}{2}\,,\; y_n + \frac{h}{2}K_2\right)$$

$$K_4 = f(t_{n+1}\,,\; y_n + h\,K_3)$$

$$\frac{d\,y(t)}{d\,t} = -a\,y(t) \quad , \quad y(0) = 1$$

$$y_{exact} = e^{-a\,t}$$

| n | 0 | 1 | 2 | 3 | 4 | 5 | 10 | 100 | 1000 |
|---|---|---|---|---|---|---|---|---|---|
| t / a | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 1.0 | 10.0 | 100.0 |
| $y_{exact}$ | 1 | 0.90484 | 0.81873 | 0.74082 | 0.67032 | 0.60653 | 0.36788 | $4.54\ 10^{-5}$ | $3.72\ 10^{-44}$ |
| $y_{Euler}$ | 1 | 0.900 | 0.810 | 0.729 | 0.656 | 0.590 | 0.349 | $2.6\ 10^{-5}$ | $1.0\ 10^{-46}$ |
| $y_{R\text{-}K\ 2}$ | 1 | 0.905 | 0.819 | 0.741 | 0.671 | 0.607 | 0.369 | $4.6\ 10^{-5}$ | $4.45\ 10^{-44}$ |
| $y_{R\text{-}K\ 4}$ | 1 | 0.90484 | 0.81873 | 0.74082 | 0.67032 | 0.60653 | 0.36788 | $4.54\ 10^{-5}$ | $3.72\ 10^{-44}$ |

# RUNGE-KUTTA METHODS

Euler's Method: $y_{n+1} = y_n + h\, f\,(t_n\,,\, y_n)$

Heun's Method $$y_{n+1} = y_n + \frac{h}{2}\left[f\,(t_n\,, y_n) + f\left(t_{n+1}\,, y_n + f\,(t_n\,, y_n\right)\right]$$

The Runge-Kutta Order 4 $$y_{n+1} = y_n + \frac{h}{6}(K_1 + 2\,K_2 + 2\,K_3 + K_4)$$

$K_1 = f\,(t_n\,,\, y_n)$ $\quad K_2 = f\,(t_n + \frac{h}{2}\,,\; y_n + \frac{h}{2}K_1)$ $\quad K_3 = f\,(t_n + \frac{h}{2}\,,\; y_n + \frac{h}{2}K_2)$

$K_4 = f\,(t_{n+1}\,,\; y_n + h\,K_3)$

h or $\Delta t$

1. How to choose?
2. Why constant and uniform?

# STABILITY OF RUNGE-KUTTA METHODS

$$\frac{dy}{dt} = \lambda y \qquad \text{IC: } y(0) = 1$$

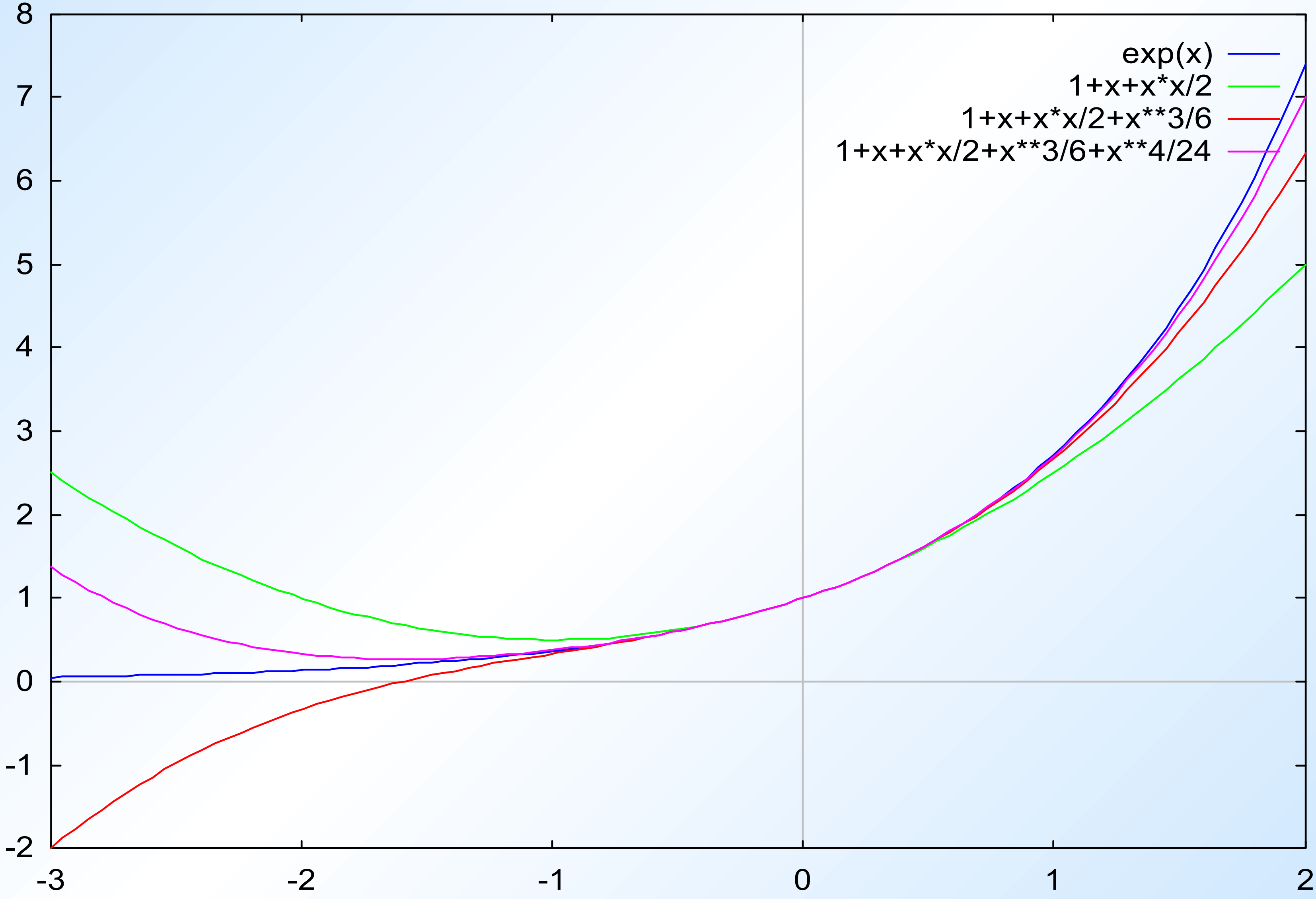
Exact Solution: $y(t) = e^{\lambda t}$

Numerical Solution: $y_{n+1} = y_n P(\lambda h)$

Euler's Method $y_{n+1} = y_n (1 + \lambda h)$

Runge-Kutta Order 2 $$y_{n+1} = y_n \left(1 + \lambda h + \frac{(\lambda h)^2}{2}\right)$$

Runge-Kutta Order 4 $$y_{n+1} = y_n \left(1 + \lambda h + \frac{(\lambda h)^2}{2} + \frac{(\lambda h)^3}{3!} + \frac{(\lambda h)^4}{4!}\right)$$


exp(x)
1+x+x*x/2
1+x+x*x/2+x**3/6
1+x+x*x/2+x**3/6+x**4/24
8
7
6
5
4
3
2
1
0
-1
-2
-3
-2
-1
0
1
2

# EXAMPLE ON STABILITY OF RUNGE-KUTTA METHODS

Given: $$\frac{dy}{dt} = 10y - 10t - 1$$

General Solution $$y(t) = A\,e^{10t} - t$$

Initial Condition $y(0) = 0$

Exact Solution: $y(t) = -t$ , $A \equiv 0$

| t | Exact | Runge-Kutta Order 4 | | |
|---|---|---|---|---|
| | | h = 0.01 | | |
| 0.2 | - 0.2 | - 0.2 | | |
| 0.4 | - 0.4 | - 0.39998 | | |
| 0.6 | - 0.6 | - 0.59986 | | |
| 0.8 | - 0.8 | - 0.79896 | | |
| 1.0 | - 1.0 | - 0.99232 | | |
| 1.2 | - 1.2 | - 0.11134 | | |
| 1.4 | - 1.4 | - 0.98318 | | |
| 1.6 | - 1.6 | 1.4702 | | |
| 1.8 | - 1.8 | 20.815 | | |
| 2.0 | - 2.0 | 164.59 | | |

| t | Exact | Runge-Kutta Order 4 | | |
|---|---|---|---|---|
| | | h = 0.01 | h = 0.001 | |
| 0.2 | - 0.2 | - 0.2 | - 0.19998 | |
| 0.4 | - 0.4 | - 0.39998 | - 0.39988 | |
| 0.6 | - 0.6 | - 0.59986 | - 0.59918 | |
| 0.8 | - 0.8 | - 0.79896 | - 0.79394 | |
| 1.0 | - 1.0 | - 0.99232 | - 0.95529 | |
| 1.2 | - 1.2 | - 0.11134 | - 0.86994 | |
| 1.4 | - 1.4 | - 0.98318 | 1.0378 | |
| 1.6 | - 1.6 | 1.4702 | 16.410 | |
| 1.8 | - 1.8 | 20.815 | 131.26 | |
| 2.0 | - 2.0 | 164.59 | 981.07 | |

| t | Exact | Runge-Kutta Order 4 | | |
|---|---|---|---|---|
| | | h = 0.01 | h = 0.001 | h = 0.0001 |
| 0.2 | - 0.2 | - 0.2 | - 0.19998 | - 0.19986 |
| 0.4 | - 0.4 | - 0.39998 | - 0.39988 | - 0.39920 |
| 0.6 | - 0.6 | - 0.59986 | - 0.59918 | - 0.59483 |
| 0.8 | - 0.8 | - 0.79896 | - 0.79394 | - 0.76311 |
| 1.0 | - 1.0 | - 0.99232 | - 0.95529 | - 0.72925 |
| 1.2 | - 1.2 | - 0.11134 | - 0.86994 | 0.79465 |
| 1.4 | - 1.4 | - 0.98318 | 1.0378 | 13.316 |
| 1.6 | - 1.6 | 1.4702 | 16.410 | 107.03 |
| 1.8 | - 1.8 | 20.815 | 131.26 | 800.08 |
| 2.0 | - 2.0 | 164.59 | 981.07 | 5921.3 |