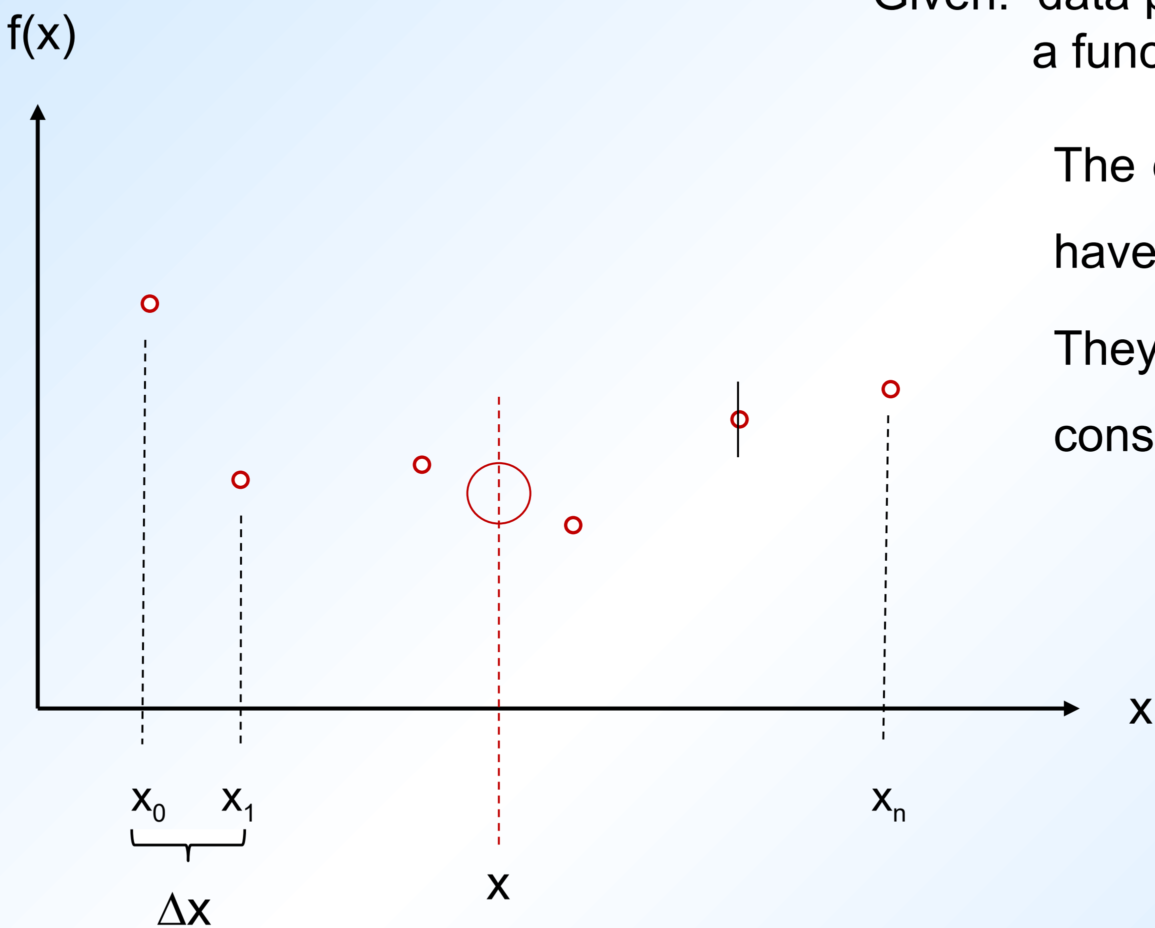
INTERPOLATION

In engineering and science, one often has a number of data points, obtained by sampling or experimentation, which represent the values of a function for a limited number of values of the independent variable. It is often required to **interpolate**, i.e., estimate the value of that function for an intermediate value of the independent variable. Remember thermodynamic tables.

A closely related problem is the approximation of a complicated function by a simple function. Suppose the formula for some given function is known, but too complicated to evaluate efficiently. A few data points from the original function can be interpolated to produce a simpler function which is still fairly close to the original. The resulting gain in simplicity may outweigh the loss from interpolation error.

See the book chapter "Fitting" on OdtuClass.



Given: data points or
a function changed to data points

The data points may or may not have statistical errors

They may or maynot have constant spacing

Questions:

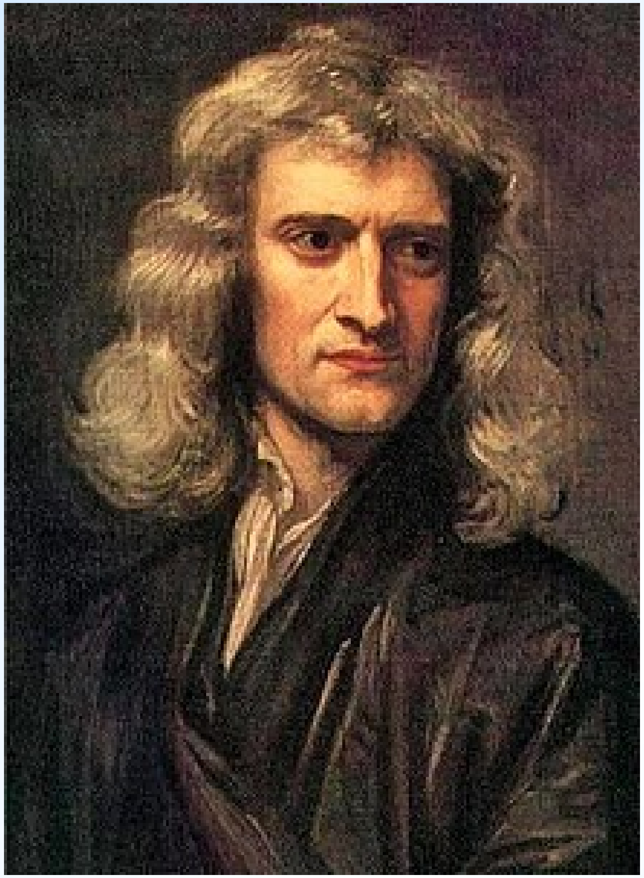
$$\left.\begin{array}{l} f(x) = ? \\[2ex] \dfrac{d^n f}{dx^n} = ? \\[2ex] \displaystyle\int_{x_0}^{x_n} f(x)\, dx = ? \end{array}\right\} \text{Error?}$$

Interpolation Schemes with Polynomials

Name	Advantages	Disadvantages
Newton-Gregory	The degree of the polynomial can be changed easily. Evaluation of errors is easy.	A difference table or a divided difference table must be prepared.
Lagrange	Convenient form Easy to program	Cumbersome for hand calculations
Spline	Applicable to any number of data points	Need to solve a set of simultaneous equations
Hermite	Accuracy is high because polynomial is fitted to derivatives also.	Need values of derivatives
Chebyshev	Errors are more evenly distributed than that with an equispaced grid.	Function should be given. Unevenly distributed grid points

Interpolation Schemes with Polynomials

Name	Advantages	Disadvantages
Newton-Gregory	The degree of the polynomial can be changed easily. Evaluation of errors is easy.	A difference table or a divided difference table must be prepared.



Isaac
Newton
1642 - 1726

James Gregory
1638 - 1675

Newton-Gregory Forward Polynomial

Define $s = \dfrac{x - x_0}{h}$ where h is constant (equally spaced data points)

$$P_n(s) = f_0 + s\,\Delta f_0 + \frac{s(s-1)}{2!}\Delta^2 f_0 + \ldots + \frac{s(s-1)\ldots(s-n+1)}{n!}\Delta^n f_0$$

Error:

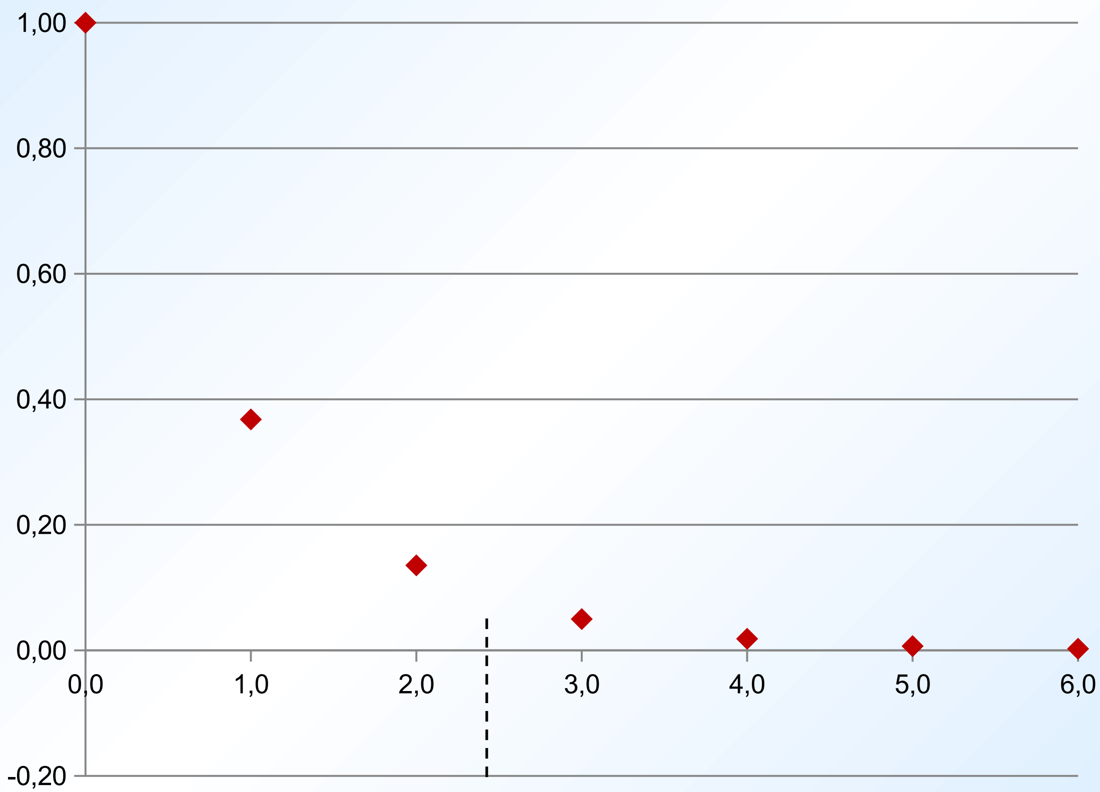
$$E(s) = \frac{s(s-1)(s-2)\ldots(s-n)}{(n+1)!} h^{n+1} f^{(n+1)}(\xi) \quad , \quad s_0 < \xi < s$$

$$E(x) = \frac{(x-x_0)(x-x_1)\ldots(x-x_n)}{(n+1)!} f^{(n+1)}(\xi) \, , \quad x_0 < \xi < x$$

To estimate the error, replace the $(n+1)^{th}$ derivative with $(n+1)^{th}$ divided difference.

Example:

x_i	0	1	2	3	4	5	6
$f(x_i)$	1	0.368	0.135	0.050	0.018	0.007	0.002



At $x = 2.5$

$f(x) = f(2.5) = ?$

Error = ?

Solution:

- Choose a polynomial of some (small) degree, $P_n(x)$;
- Decide on the range (the data points to be used);
- Find the constants of the polynomial;
- Evaluate the polynomial at $x = 2.5$. This is the interpolation;
- Estimate the error.

Questions: How do you find the constants, C_i's?

$$P_n(x) = C_0 + C_1 x + C_2 x^2 + ... + C_n x^n$$

and, how do you find (or estimate) the error?

Answer: Use Newton-Gregory difference tables

Difference Table for f(x) = e^{-x}

x	f (x)	Δ f	Δ^2 f	Δ^3 f	Δ^4 f
0	1				
		- 0.632120559			
1	0.367879441		0.399576401		
		- 0.232544158		- 0.252580458	
2	0.135335283		0.146995943		0.1596613
		- 0.085548215		- 0.092919158	
3	0.049787068		0.054076785		0.05873611
		- 0.031471429		- 0.034183048	
4	0.018315639		0.019893738		0.021607807
		- 0.011577692		- 0.012575241	
5	0.006737947		0.007318497		
		- 0.004259195			
6	0.002478752				

Example: At $x = 2.5$ $f(x) = f(2.5) = ?$

Choose a polynomial of some degree, $P_n(x)$: Say, $n = 3$

Which four points? Approximately symmetrical around the point of interest

Find the constants of the polynomial: Use the difference table and the formula

Evaluate the polynomial at $x = 2.5$. This is the interpolation.

How do you estimate the error? Use the difference table and the formula

N-G Polynomial Interpolation with Unequally-spaced Data

$$P_n(x) = f(x_i) + D f_i (x - x_i) + D^2 f_i (x - x_i)(x - x_{i+1}) + \dots + D^n f_i (x - x_i)(x - x_{i+1}) \dots (x - x_{i+n-1})$$

Error: $$E(x) = f(x) - P_n(x) = \prod_{i=0}^{n} (x - x_i) \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

To estimate the error, replace the $(n+1)^{th}$ derivative with $(n+1)^{th}$ divided difference.

$$E(x) \cong \prod_{i=0}^{n} (x - x_i)\ D^{n+1} f_i$$

Divided Difference Table for f (x) = e^{-x}

x	f (x)	D f	D^2 f	D^3 f
0	1			
		-0.632120559		
1	0.367879441		0.157691457	
		-0.159046186		-0.021506018
3	0.049787068		0.02865535	
		-0.015769439		
6	0.002478752			

Newton-Gregory Forward Polynomial:

$$P_3(x) = f(x_0) + (x - x_0)\,Df_0 + (x - x_0)(x - x_1)\,D^2 f_0 + (x - x_0)(x - x_1)(x - x_2)\,D^3 f_0$$

$$P_3(x) = 1 + x\left(-0.6321206\right) + x(x-1)\left(0.15769146\right) + x(x-1)(x-3)\left(-0.0215060\right)$$

Error:

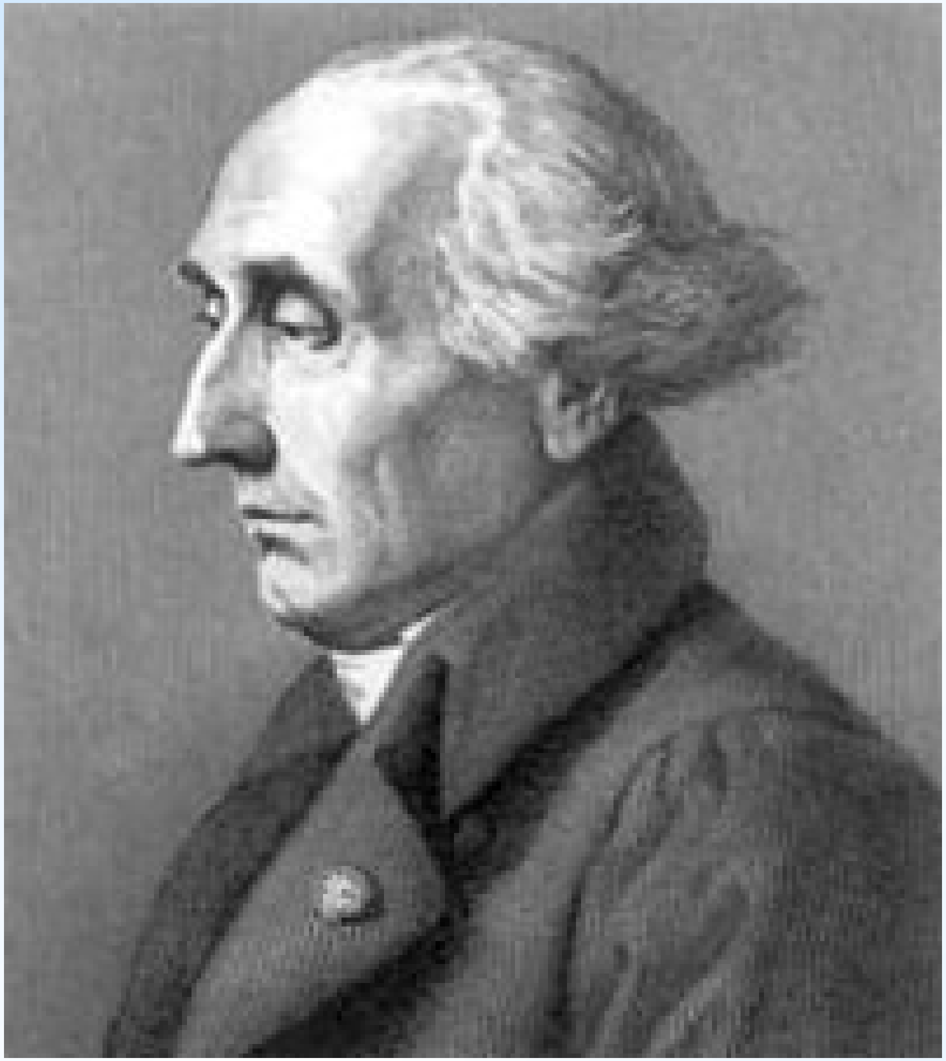
$$E(x) = \frac{(x-1)(x-2)\left(x-3\right)(x-4)}{4!}\,f^{(4)}(\xi)$$

$$E(x) \cong (x-1)(x-2)\left(x-3\right)(x-4)\,D^4 f_0$$

Note that the fourth divided difference, D^4f, does nor exists. So, what do you do?

Interpolation Schemes with Polynomials

Name	Advantages	Disadvantages
Newton-Gregory	The degree of the polynomial can be changed easily. Evaluation of errors is easy.	A difference table or a divided difference table must be prepared.
Lagrange	**Convenient form** **Easy to program**	**Cumbersome for hand calculations**



Joseph-Louis Lagrange

French (originally Italian) mathematician

1736 - 1813

Polynomial Interpolation with Unequally-spaced Data

Lagrange Form of a Polynomial:

$$
\begin{aligned}
P_n(x) = {} & \frac{(x - x_1)(x - x_2)\ldots(x - x_{n-1})(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_{n-1})(x_0 - x_n)}\, f(x_0) \\
& + \frac{(x - x_0)(x - x_2)\ldots(x - x_{n-1})(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_{n-1})(x_1 - x_n)}\, f(x_1) \\
& + \ldots \\
& + \frac{(x - x_0)(x - x_1)\ldots(x - x_{n-2})(x - x_n)}{(x_{n-1} - x_0)(x_{n-1} - x_1)\ldots(x_{n-1} - x_{n-2})(x_{n-1} - x_n)}\, f(x_{n-1}) \\
& + \frac{(x - x_0)(x - x_1)\ldots(x - x_{n-2})(x - x_{n-1})}{(x_n - x_0)(x_n - x_1)\ldots(x_n - x_{n-2})(x_n - x_{n-1})}\, f(x_n)
\end{aligned}
$$

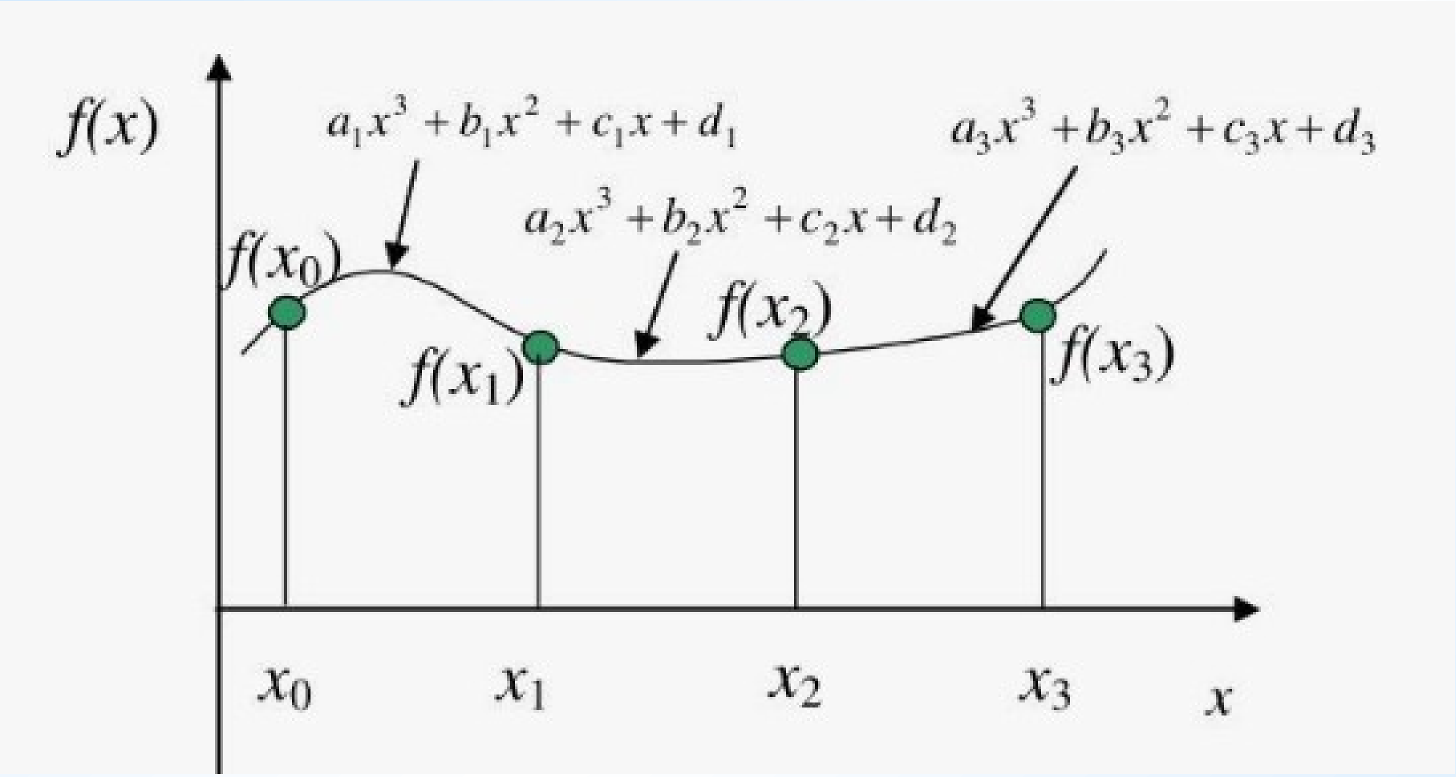
Interpolation Schemes with Polynomials

Name	Advantages	Disadvantages
Newton-Gregory	The degree of the polynomial can be changed easily. Evaluation of errors is easy.	A difference table or a divided difference table must be prepared.
Lagrange	Convenient form Easy to program	Cumbersome for hand calculations
Spline	**Applicable to any number of data points**	**Need to solve a set of simultaneous equations**

Pierre Étienne Bezier

French Engineer

1910 - 1999



Pass a third-degree polynomial through every interval, equating functional values as well as the first and second derivatives at the intermediate junctions.

Cubic Spline Fitting

$$P_{3,k}(x) = a_k\,(x - x_i)^3 + b_k\,(x - x_i)^2 + c_k\,(x - x_i) + d_k$$

where k is the interval number, $k = 1, 2, \ldots, N-1$

x_i is the beginning point of each interval

N is the total number of points

Note the following:

There are $N - 1$ such equations, one for each interval, k, whose coefficients are to be found.

- The total number of unknown coefficients is $4\,(N - 1)$.
- There must be $4\,(N - 1)$ equations to determine all the unknown parameters (coefficients of the cubic polynomials).

Cubic Spline Coefficients

Based on the second derivatives, the recurrence formula for the second derivative at each node i is

$$h_k\, y_i'' + 2\,(h_k + h_{k+1})\, y_{i+1}'' + h_{k+1}\, y_{i+2}'' = 6\left(\frac{y_{i+2} - y_{i+1}}{h_{k+1}} - \frac{y_{i+1} - y_i}{h_k}\right)$$

where i = 1, 2, .. , N - 2, and h_k is the interval size for each interval, k.

The second derivatives at i = 1 and i = N are taken from the end conditions.

Once all the second derivatives are determined, the coefficients of the polynomials at each interval k are found:

$$a_k = \frac{y_{i+1}'' - y_i''}{6\,h_k} \qquad b_k = \frac{y_i''}{2} \qquad c_k = \frac{y_{i+1} - y_i}{h_k} - \frac{2\,h_k\, y_i'' + h_k\, y_{i+1}''}{6} \qquad d_k = y_i$$

Interpolation Schemes with Polynomials

Name	Advantages	Disadvantages
Newton-Gregory	The degree of the polynomial can be changed easily. Evaluation of errors is easy.	A difference table or a divided difference table must be prepared.
Lagrange	Convenient form Easy to program	Cumbersome for hand calculations
Spline	Applicable to any number of data points.	Need to solve a set of simultaneous equations
Hermite	**Accuracy is high because polynomial is fitted to derivatives also.**	**Need values of derivatives**

Charles Hermite

French mathematician

1822 - 1901

Hermite Fitting

For standard polynomial interpolation problems, we seek to satisfy conditions of the form

$$f(x_i) \cong P(x_i)$$

If all we know is function values, this is a reasonable approach. But sometimes we have more information. Hermite interpolation constructs an interpolant based not only on equations for the function values, but also for the derivatives.

As with polynomial interpolation based just on function values, we can express the polynomial that satisfies the conditions with respect to several different bases: monomial, Lagrange, or Newton.

Monomial form: $$P(x) = C_0 + C_1\, x + C_2\, x^2 + C_3\, x^3 + \ldots$$

Example: Find a polynomial that satisfies prescribed conditions on the values and derivatives at the end points of the interval [−1, 1]. That is, we require

$$P(1) = f(1) \qquad P(-1) = f(-1)$$

$$P'(1) = f'(1) \qquad P'(-1) = f'(-1)$$

Since there are four conditions to satisfy, the polynomial is of third degree:

$$P_3(x) = C_0 + C_1 x + C_2 x^2 + C_3 x^3$$

This gives us the relation:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & -2 & 3 \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} f(1) \\ f(-1) \\ f'(1) \\ f'(-1) \end{pmatrix}$$

Solve for the coefficients.

Interpolation Schemes with Polynomials

Name	Advantages	Disadvantages
Newton-Gregory	The degree of the polynomial can be changed easily. Evaluation of errors is easy.	A difference table or a divided difference table must be prepared.
Lagrange	Convenient form Easy to program	Cumbersome for hand calculations
Spline	Applicable to any number of data points	Need to solve a set of simultaneous equations.
Hermite	Accuracy is high because polynomial is fitted to derivatives also.	Need values of derivatives.
Chebyshev	**Errors are more evenly distributed than that with an equispaced grid.**	**Function should be given. Unevenly distributed grid points**

Pafnuty Lvovich Chebyshev

(Пафну́тий Льво́вич Чебышёв)

Russian mathematician

1821 - 1894

COMPUTATIONAL PROCEDURE

Frequently, several methods are available for the numerical solution of a given mathematical problem. Few relevant criteria for selection of a method:

- Accuracy
- Efficiency
- Numerical stability
- Programming simplicity
- Versatility
- Computer storage requirements
- Interfacing with available software
- Previous experience with a given method

Norms

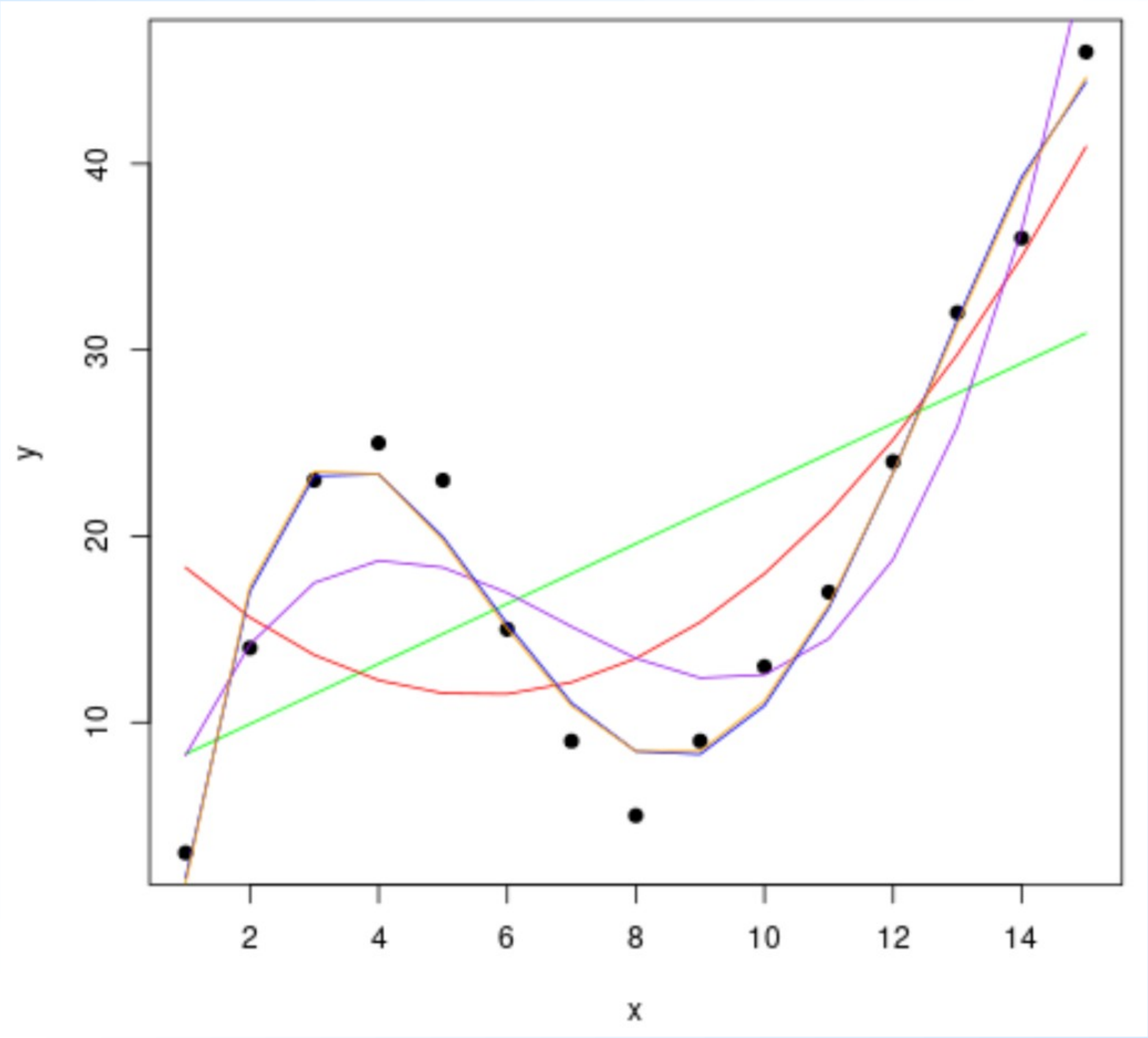
Given: A function, f(x), or tabulated data points in a certain range, [a , b]

f(x) is real-valued, well-behaving, but complicated.

Problem: Find one, simple(r) function, g(x), that represents, or fits, the given f(x) or the data points in [a , b] according to a certain "goodness" or "closeness" required.

How do we define "goodness" of fitting?

Answer: There are basically three ways of defining "goodness" which are called "Norms“.


y
40
30
20
10
2
4
6
8
10
12
14
x

Least Deviation or L_1 Norm: Minimize average error, i.e.,

$$\int_a^b \left| f(x) - g(x) \right| \, dx$$

Least Squares or L_2 Norm: Minimize sum of the squares of the error, i.e.

$$\int_a^b \left[f(x) - g(x) \right]^2 \, dx$$

Chebyshev or L_∞ Norm: Minimize maximum error, i.e.,

Max. $\left| f(x) - g(x) \right|$ in [a , b]

The meaning of the **Least-Squares or L_2 Norm** is intuitively not as clear as the others, but it is very widely used in many problems because of its simplicity in implementation and analysis, even when the other norms are more natural and meaningful.

L_1 Norm is more difficult to use and analyze. Its main attraction is in data analysis and smoothing as it is remarkably insensitive to random errors, wild points, and other uncertainties.

Chebyshev or L_∞ Norm is primarily used for approximating standard mathematical functions rather than data points, because it is unsuitable when uncertainty is present in the data. It often magnifies the effect of a single uncertainty or random error.

ORTHOGONAL FUNCTIONS

Two functions f(x) and g(x) are orthogonal with respect to a weight(ing) function w(x) on an interval [a , b] if

$$\int_a^b w(x)\, f(x)\, g(x)\, dx = 0$$

Example: sin(nπx) and sin(mπx), m ≠ n, are orthogonal with respect to w(x) = 1 over the interval [0,1].

$$\int_0^1 \sin(n\pi x)\,\sin(m\pi x)\, dx = 0 \quad , \quad m \neq n$$

ORTHOGONAL FUNCTIONS

Trigonometric Functions

- Sines
- Cosines

Polynomials

- Legendre polynomials
- Laguerre polynomials
- Hermite polynomials
- Chebyshev polynomials
- Zernike polynomials
- Others?

Binary–valued functions

- Walsh fuctions
- Haar wavelets

Rational functions

- Legendre rational functions
- Chebyshev rational functions

Legendre Orthogonal Polynomials

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}\left(3\,x^2 - 1\right)$$

$$P_3(x) = \frac{1}{2}\left(5\,x^3 - 3\,x\right)$$

$$P_4(x) = \frac{1}{8}\left(35\,x^4 - 30\,x^2 + 3\right)$$

$$P_5(x) = \frac{1}{8}\left(63\,x^5 - 70\,x^3 + 15\,x\right)$$

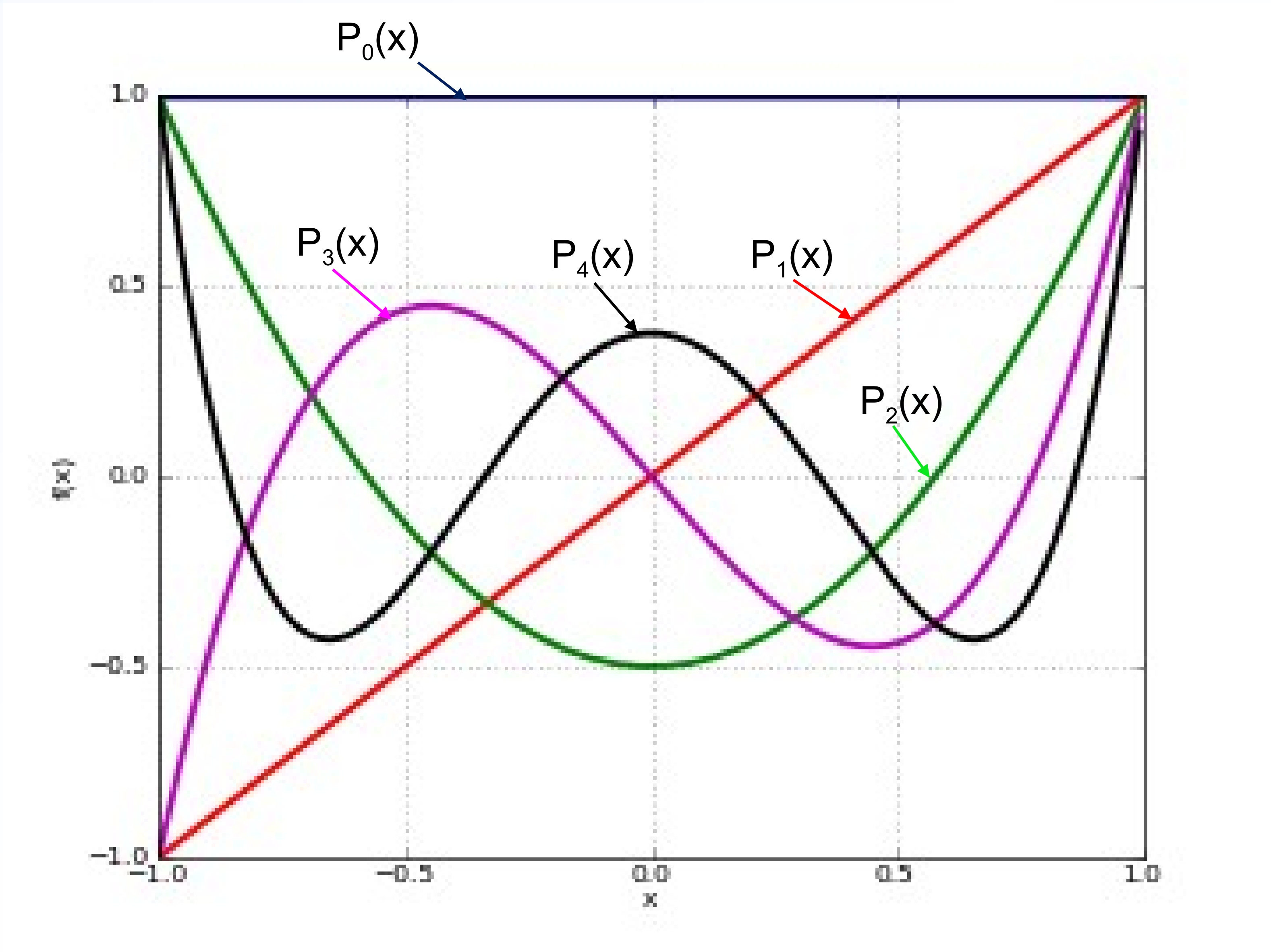
Orthogonality Property

$$\int_{-1}^{1} (1)\,P_n(x)\,P_m(x)\,dx = 0 \quad \text{if } n \neq m$$

$$\int_{-1}^{1} (1)\,P_n(x)\,P_n(x)\,dx = \frac{2}{2\,n + 1}$$

range of orthogonality [-1,+1]

weighting function = 1



$P_0(x)$
$P_3(x)$
$P_4(x)$
$P_1(x)$
$P_2(x)$
1.0
0.5
0.0
−0.5
−1.0
f(x)
−1.0
−0.5
0.0
0.5
1.0
x

ORTHOGONAL FUNCTIONS

Theorem: A function f(x) can be expressed as a linear combination of a complete set of mutually orthogonal functions

If $Q_n(x)$, n = 1, 2, 3, ..., form a complete orthogonal set of functions, then

$$f(x) = a_1\, Q_1(x) + a_2\, Q_2(x) + \ldots$$

Where the coefficients can be determined by

$$a_n = \frac{\int_a^b w(x)\, Q_n(x)\, f(x)\, dx}{\int_a^b w(x)\, \left(Q_n(x)\right)^2\, dx}$$

ORTHOGONAL FUNCTIONS

Example: Given the step function

$$f(x) = \begin{cases} -1 & -1 < x < -\frac{1}{2} \\ +1 & -\frac{1}{2} < x < \frac{1}{2} \\ -1 & \frac{1}{2} < x < 1 \end{cases}$$

Expand f (x) in terms of the cosine orthogonal functions:

$$f(x) = \sum_{n=0}^{\infty} a_n \cos(n\pi x)$$

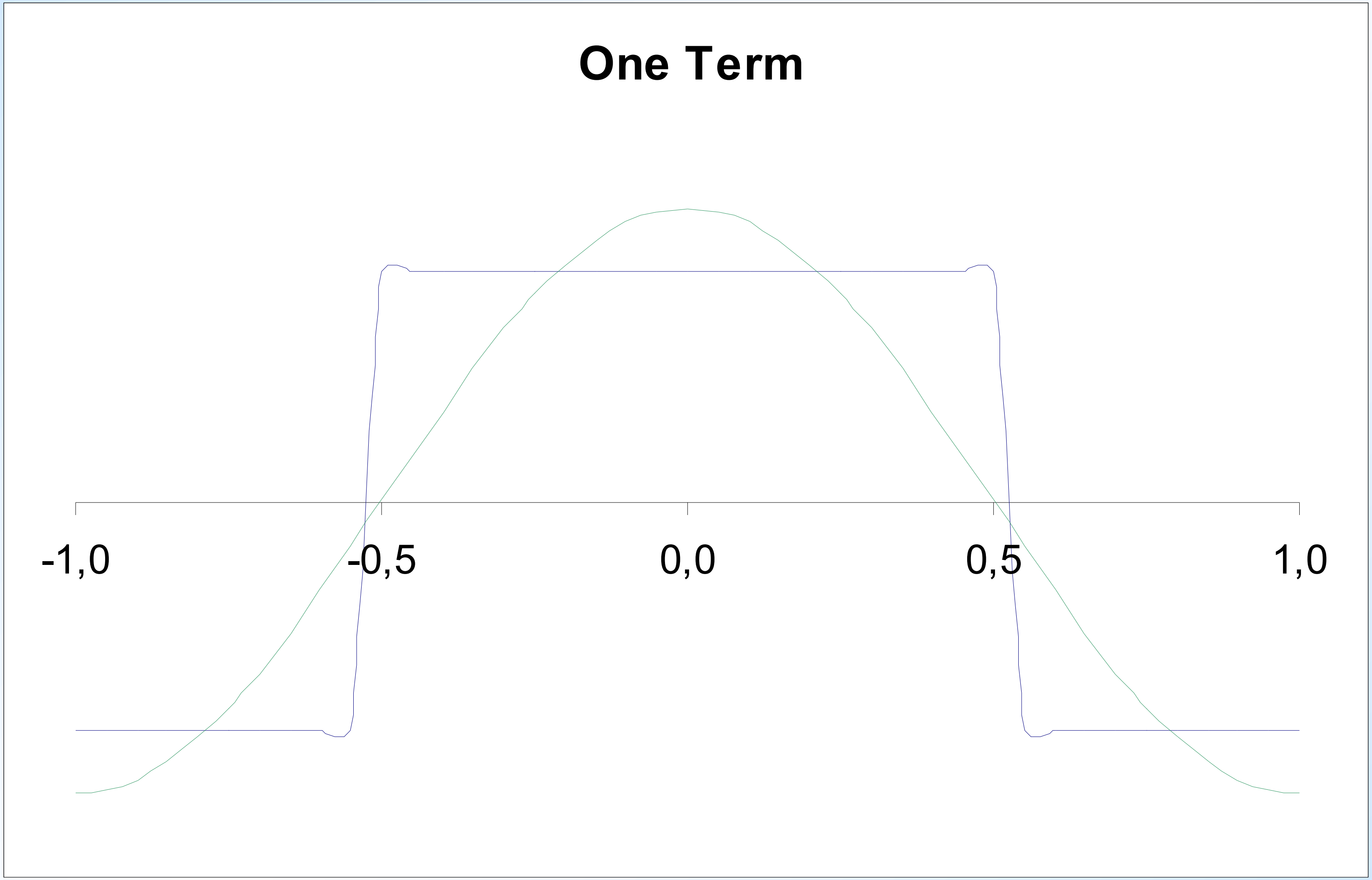
ORTHOGONAL FUNCTIONS

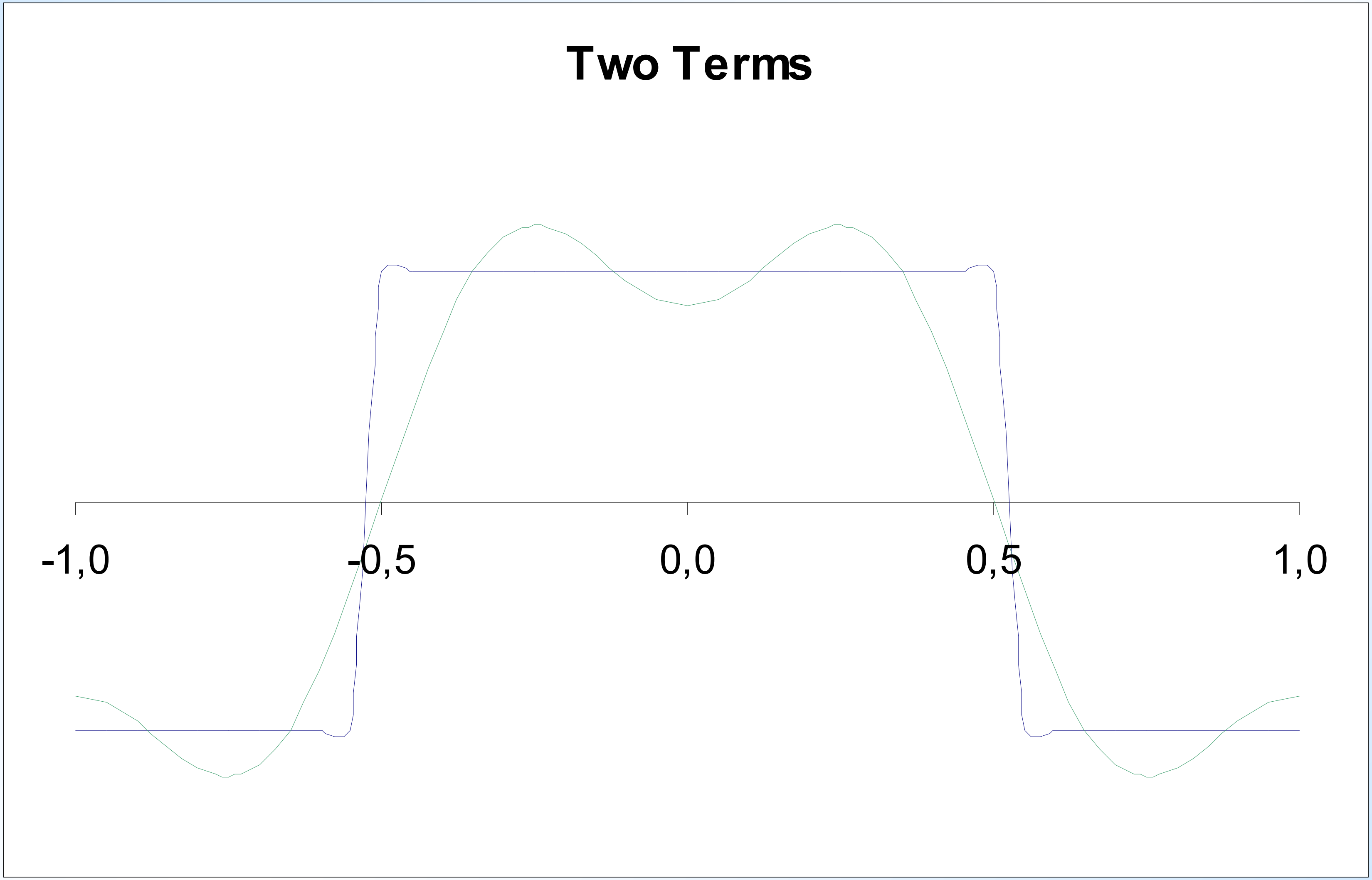
The infinite set of cosines form an othogonal set in [0,1] with w(x) = 1

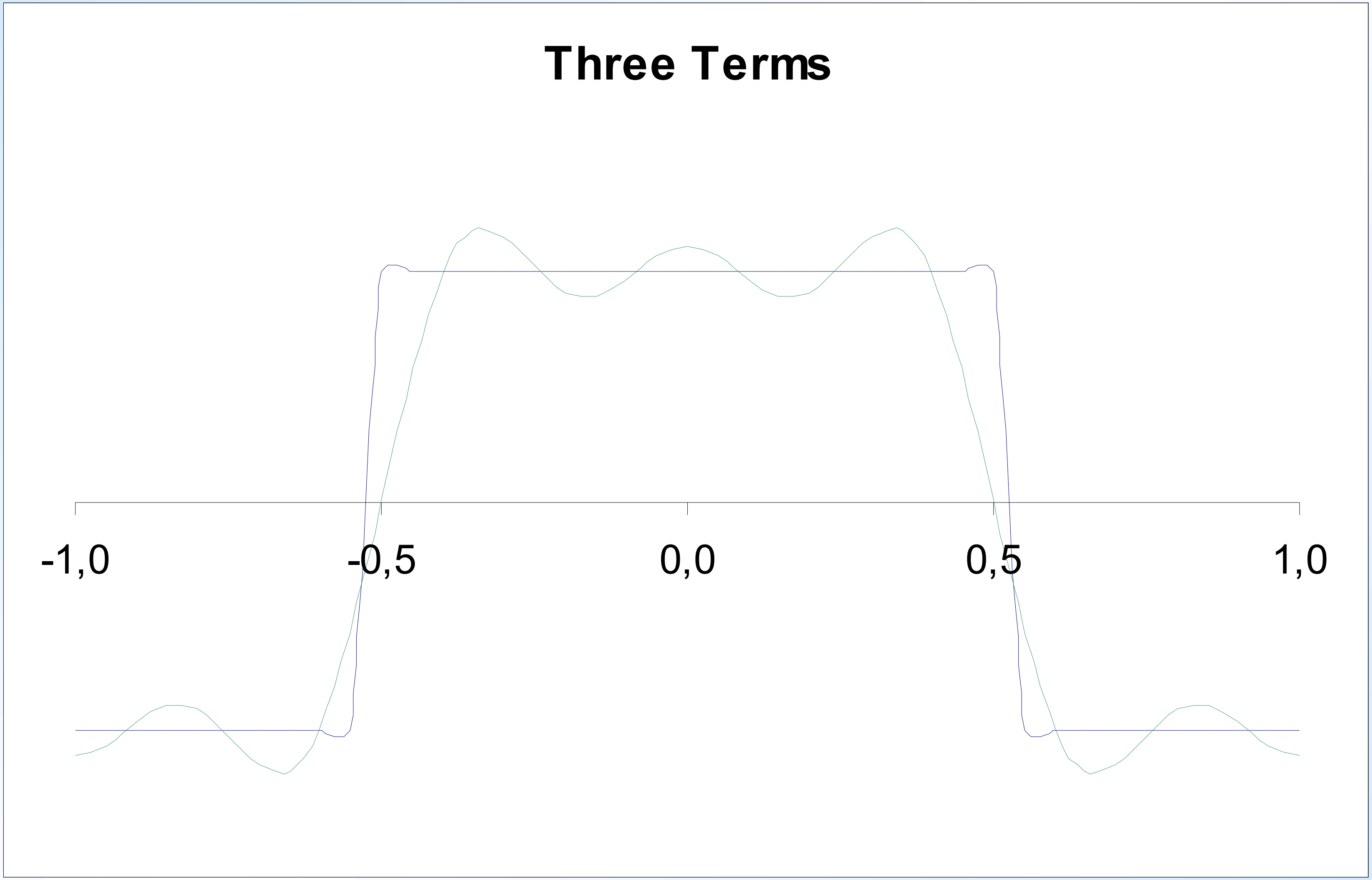
$$\int_0^1 \cos(n\pi x)\cos(m\pi x)\,dx = \begin{cases} 0 & m \neq n \\ \frac{1}{2} & m = n \end{cases}$$

Calculate the coefficients of Fourier cosine series, a_n's, for n = 0, 1, ...

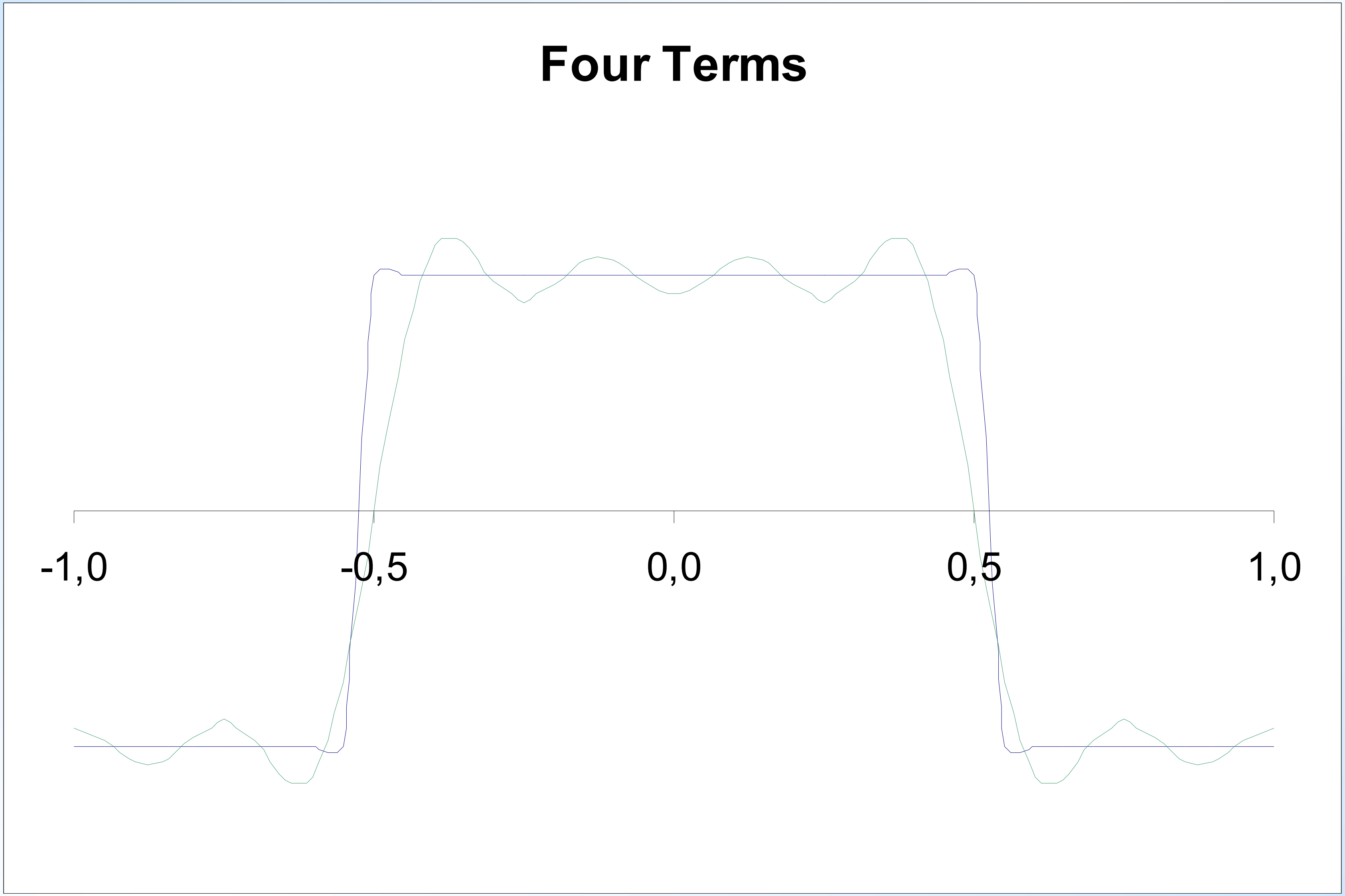
$$f(x) = \frac{4}{\pi}\cos(\pi x) - \frac{4}{3\pi}\cos(3\pi x) + \frac{4}{5\pi}\cos(5\pi x) - \frac{4}{7\pi}\cos(7\pi x) + \ldots$$


One Term
-1,0
-0,5
0,0
0,5
1,0


Two Terms
-1,0
-0,5
0,0
0,5
1,0



Three Terms
-1,0
-0,5
0,0
0,5
1,0


Four Terms
-1,0
-0,5
0,0
0,5
1,0

Orthogonality Conditions for Sine and Cosine Functions

$$\int_{-L}^{L} \cos\left(\frac{m\,\pi\,x}{L}\right)\cos\left(\frac{n\,\pi\,x}{L}\right) dx = \begin{cases} 2L & \text{if } n = m = 0 \\ L & \text{if } n = m \neq 0 \\ 0 & \text{if } n \neq m \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{m\,\pi\,x}{L}\right)\sin\left(\frac{n\,\pi\,x}{L}\right) dx = \begin{cases} L & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{m\,\pi\,x}{L}\right)\cos\left(\frac{n\,\pi\,x}{L}\right) dx = 0 \quad \text{if } m \text{ and } n \text{ are any positive integer}$$

Orthogonality Conditions for Sine and Cosine Functions

$$\int_0^L \cos\left(\frac{m\,\pi\,x}{L}\right)\cos\left(\frac{n\,\pi\,x}{L}\right)dx = \begin{cases} L & \text{if } n = m = 0 \\ L/2 & \text{if } n = m \neq 0 \\ 0 & \text{if } n \neq m \end{cases}$$

$$\int_0^L \sin\left(\frac{m\,\pi\,x}{L}\right)\sin\left(\frac{n\,\pi\,x}{L}\right)dx = \begin{cases} L/2 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

CHEBYSHEV (*TCHEBYCHEV*) POLYNOMIALS

$$T_0(x) = 1$$

$$T_1(x) = x$$

$$T_2(x) = 2x^2 - 1$$

$$T_3(x) = 4x^3 - 3x$$

$$T_4(x) = 8x^4 - 8x^2 + 1$$

$$T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x)$$

$$T_n(\cos\theta) = \cos(n\,\theta) \quad , \quad n = 0, 1, 2, 3, \ldots$$

$$T_0(x) = \cos(0\,\theta) = 1$$

$$T_1(x) = \cos(1\theta) = x$$

$$T_2(x) = \cos(2\,\theta) = 2\cos(\theta)\cos(\theta) - 1 = 2x^2 - 1$$

CHEBYSHEV POLYNOMIALS

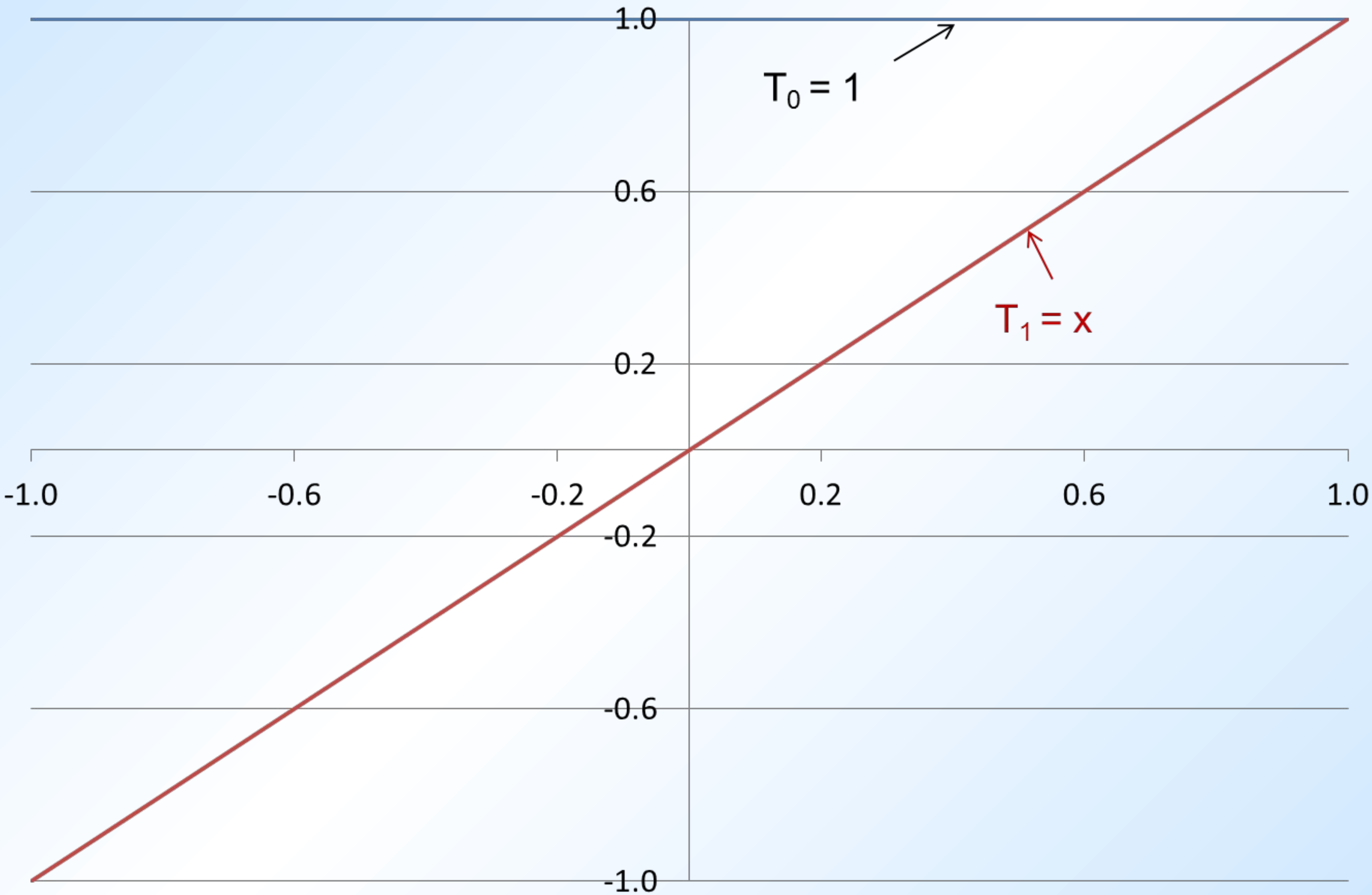
Orthogonality Property:

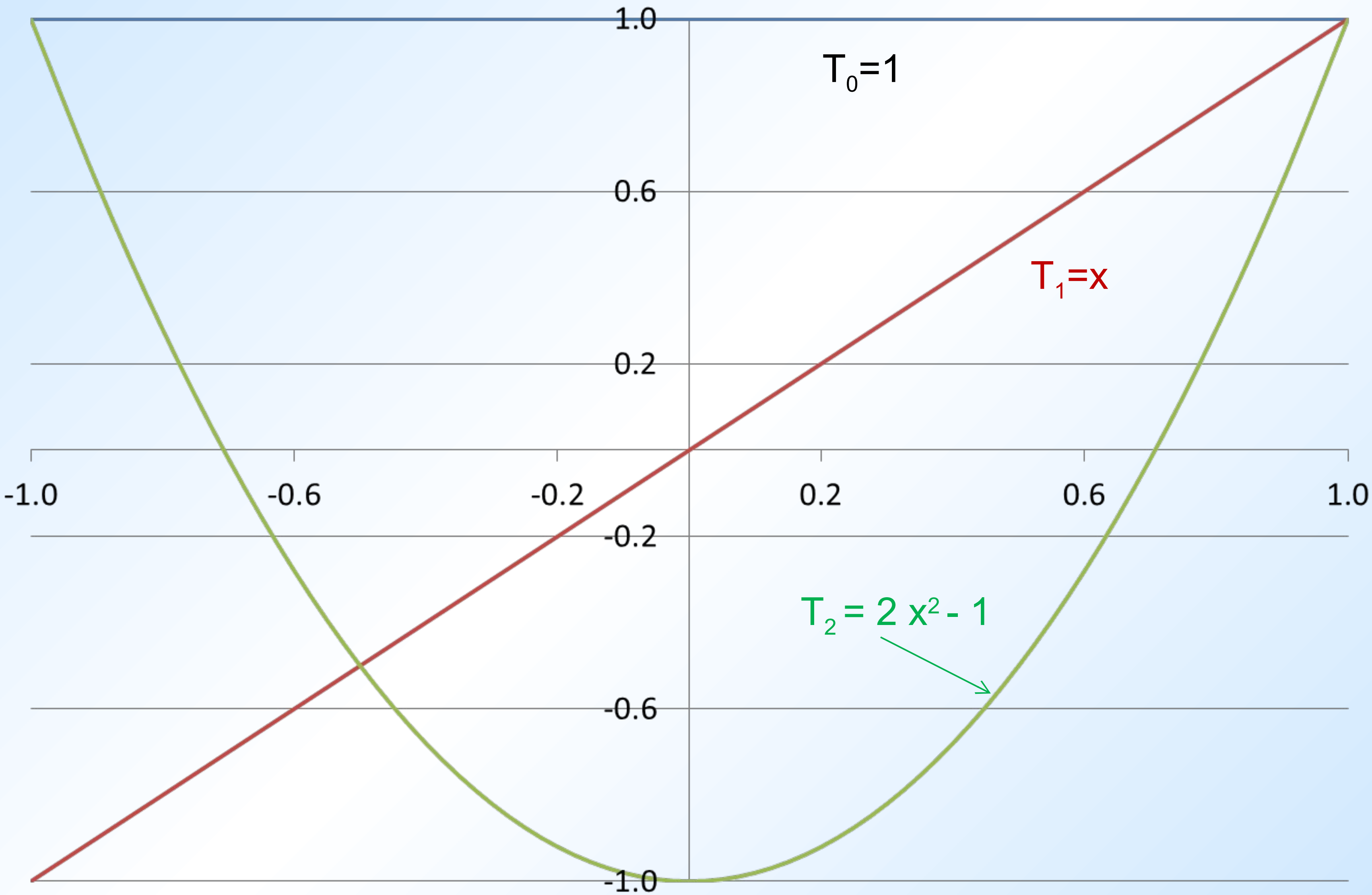
$$\int_{-1}^{1} \frac{T_n(x)\,T_m(x)}{\sqrt{1-x^2}}\,dx = \begin{cases} 0 & n \neq m \\ \pi & n = m = 0 \\ \dfrac{\pi}{2} & n = m \neq 0 \end{cases}$$

$$f(x) = \frac{a_0}{2} + \sum_{i=1}^{\infty} a_i\, T_i(x)$$

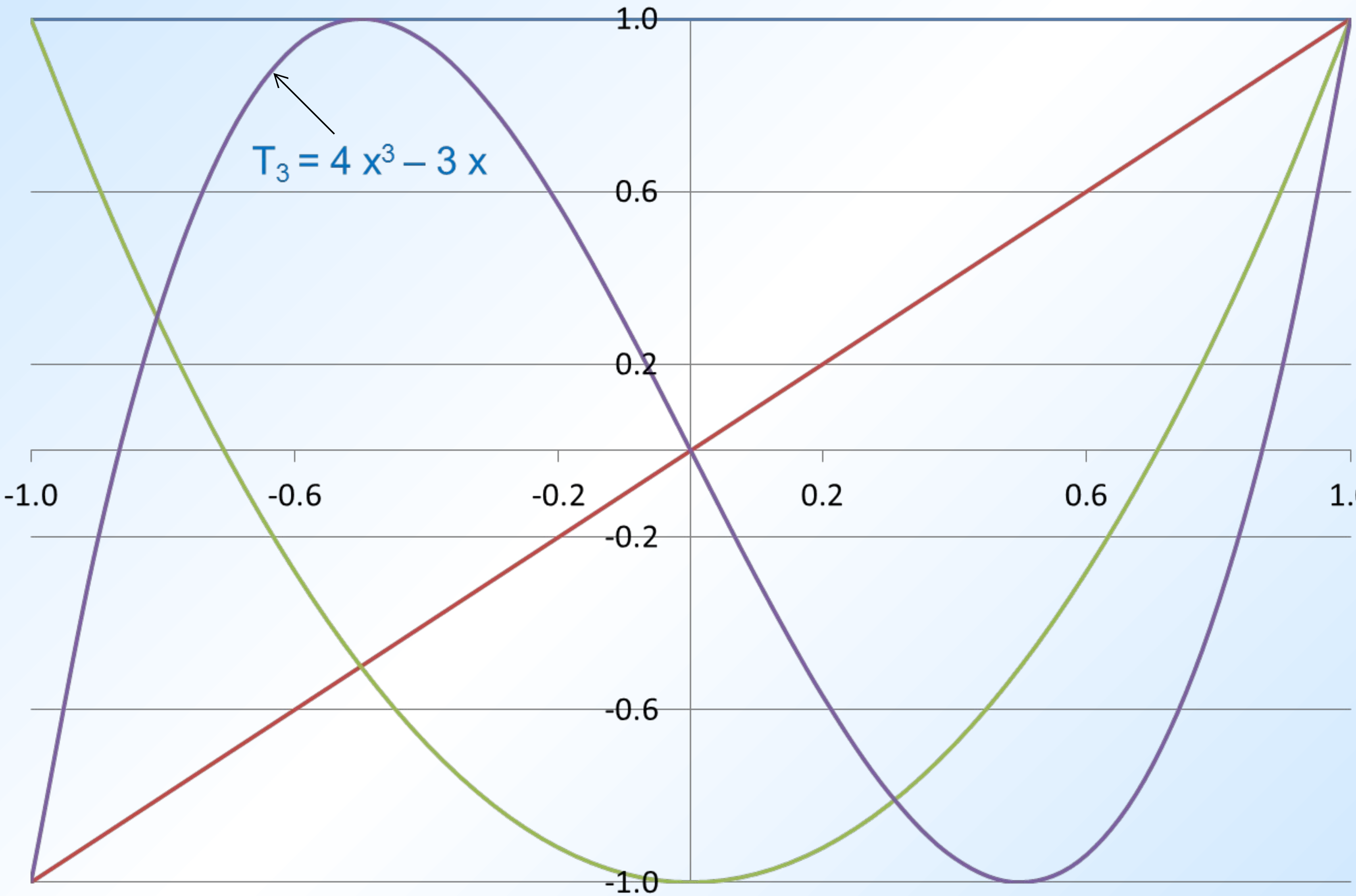
where

$$a_i = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x)\,T_i(x)}{\sqrt{1-x^2}}\,dx$$

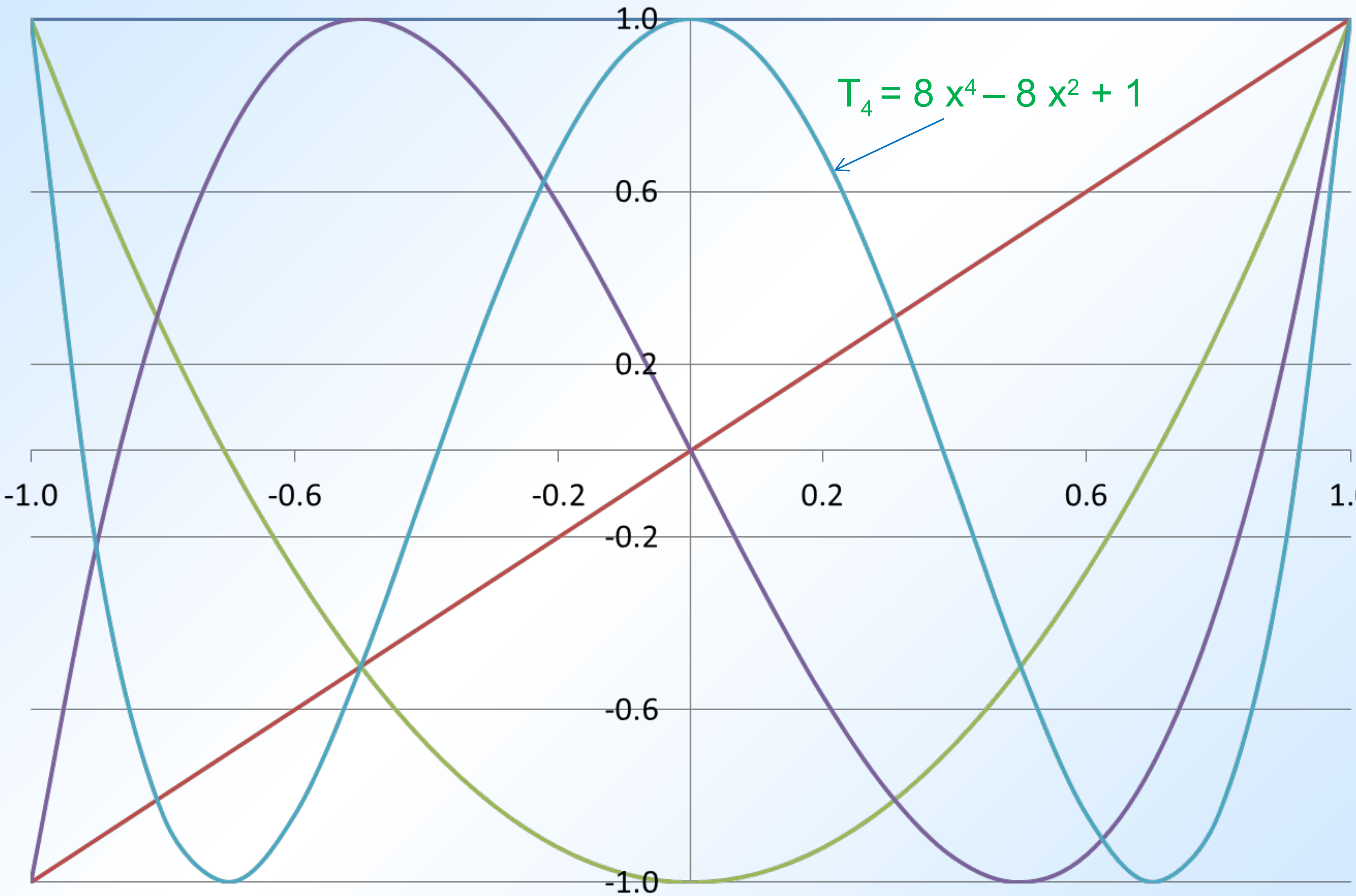

1.0
0.6
0.2
-0.2
-0.6
-1.0
-1.0
-0.6
-0.2
0.2
0.6
1.0
$T_0 = 1$
$T_1 = x$



1.0
0.6
0.2
-0.2
-0.6
-1.0
-1.0
-0.6
-0.2
0.2
0.6
1.0
T_0=1
T_1=x
T_2 = 2 x^2 - 1



$T_3 = 4\,x^3 - 3\,x$
1.0
0.6
0.2
-0.2
-0.6
-1.0
-1.0
-0.6
-0.2
0.2
0.6
1.0



$T_4 = 8x^4 - 8x^2 + 1$
1.0
0.6
0.2
-0.2
-0.6
-1.0
-1.0
-0.6
-0.2
0.2
0.6
1.0

Chebyshev Fitting

Given: f (x) in [a , b]

Problem: Find the best fitting, single polynomial with small degree, $P_n(x)$, that represents f (x) in the whole range, [a , b].

Solution:

Change variable from x in [a , b] to z in [-1 ,+1], and f(x) to F(z) :

$$z = \frac{2x - a - b}{b - a} \quad \Rightarrow \quad x = \frac{(b - a)\, z + a + b}{2} \quad , \quad dx = \frac{(b - a)}{2}\, dz$$

Minimize E (z) :
$$E(z) = F(z) - P_n(z) = \frac{(z - z_0)(z - z_1)\ldots(z - z_n)}{(n+1)!}\, F^{(n+1)}(\xi)$$

$$E(z) = \prod_{i=0}^{n} (z - z_i) \frac{F^{(n+1)}(\xi)}{(n+1)!}$$

Minimize E(z) means to find $z_0, z_1, \ldots, z_n$, which are the cross over points between F(z) and $P_n(z)$ where the error is exactly zero, such that the product, $(z - z_0)(z - z_1) \ldots (z - z_n)$ is minimized.

Chebyshev proved the following:

The roots of the (n+1)st Chebyshev polynomial, $T_{n+1}(z)$, gives the cross-over (z) points:

$$T_{n+1}(z) = (z - z_0)(z - z_1) \ldots (z - z_n) = 0$$

$$|T_{n+1}(z)| = |(z - z_0)(z - z_1) \ldots (z - z_n)| \approx 2^{-n}$$

$$\left|E(z)\right| \leq \frac{\text{Max}\left|F^{(n+1)}(\xi)\right|}{2^{n}\,(n+1)!}$$

The required fitting polynomial, $P_n(z)$ is the n-degree expansion of the function, F(z), in Chebyshev series

$$F(z) \approx P_n(z) = a_0\, T_0(z) + a_1\, T_1(z) + a_2\, T_2\,(z) + \ldots + a_n\, T_n\,(z)$$

There are three ways of finding the polynomial, $P_n(z)$:

1 Find the coefficients of the Chebyshev series expansion of F(z)

2. Find the coefficients of $P_n(z)$ that passes through the points $z_0, z_1, \ldots, z_n$

3. Use MacLaurin series expansion of F(z) and Chebyshev economization

Chebyshev Economization

Problem: Find a three-term approximation for f(x) = cos(x) on [-1,1] in the form

$\cos(x) = a + b\,x^2 + c\,x^4$

Solution:

Maclaurin Expansion: $$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots$$

Rewrite in terms of Chebyshev Polynomials using reciprocal relations:

$\cos(x) = 0.7652\,T_0 - 0.22981\,T_2 + 0.00495\,T_4 - 4.185\,10^{-5}\,T_6 + 1.938\,T_8$

Drop last two terms (small magnitude)

RECIPROCAL RELATIONS

$$1 = T_0(x)$$

$$x = T_1(x)$$

$$x^2 = \frac{1}{2}\left(T_0 + T_2\right)$$

$$x^3 = \frac{1}{4}\left(3\,T_1 + T_3\right)$$

$$x^4 = \frac{1}{8}\left(3\,T_0 + 4\,T_2 + T_4\right)$$

$$x^5 = \frac{1}{16}\left(10\,T_1 + 5\,T_3 + T_5\right)$$

$$x^6 = \frac{1}{32}\left(10\,T_0 + 15\,T_2 + 6\,T_4 + T_6\right)$$

$$f(x) = \frac{a_0}{2} + \sum_{i=1}^{\infty} a_i\,T_i(x)$$

where

$$a_i = \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)\,T_i(x)}{\sqrt{1-x^2}}\,dx$$

and transform back to power series:

$\cos(x) \approx 0.99996 - 0.49924\, x^2 + 0.03963\, x^4$

Compare at x = 1:

MacLaurin Expansion:

$f(x) \approx 0.54167$

Economized Series:

$f(x) \approx 0.54035$

$$
\begin{aligned}
T_0(x) &= 1 \\
T_1(x) &= x \\
T_2(x) &= 2x^2 - 1 \\
T_3(x) &= 4x^3 - 3x \\
T_4(x) &= 8x^4 - 8x^2 + 1 \\
T_5(x) &= 16x^5 - 20x^3 + 5x \\
T_6(x) &= 32x^6 - 48x^4 + 18x^2 - 1 \\
T_7(x) &= 64x^7 - 112x^5 + 56x^3 - 7x
\end{aligned}
$$

$$
\begin{aligned}
T_0(x) &= 1 \\
T_1(x) &= x \\
\frac{1}{2}\{T_2(x)+T_0(x)\} &= x^2 \\
\frac{1}{4}\{T_3(x)+3T_1(x)\} &= x^3 \\
\frac{1}{8}\{T_4(x)+4T_2(x)+3T_0(x)\} &= x^4 \\
\frac{1}{16}\{T_5(x)+5T_3(x)+10T_1(x)\} &= x^5 \\
\frac{1}{32}\{T_6(x)+6T_4(x)+15T_2(x)+10T_0(x)\} &= x^6 \\
\frac{1}{64}\{T_7(x)+7T_5(x)+21T_3(x)+35T_1(x)\} &= x^7
\end{aligned}
$$

Example on Chebyshev Fitting

Problem: Given the function $f(x) = \exp(-2x)$ in the range [0,1], find the first-degree polynomial, $P_1(x)$, that fits $f(x)$ according to Chebyshev (L_∞) norm.

Solution 1: Using Orthogonality of Chebyshev Polynomial:

Expansion of $f(x)$ in Chebyshev orthogonal polynomials gives

$$f(x) \approx P_1(x) = a_0\, T_0(x) + a_1\, T_1(x)$$

where $T_0(x) = 1$ and $T_1(x) = x$

There are two ways to determine the coefficients a_0, and a_1. The exact procedure is to use the orthogonality property of the Chebyshev functions and evaluate the appropriate integrals.

The first step is to re-define the independent variable, x, such that the range of interest is the range of orthogonality of the Chebyshev functions:

$$z = \frac{2x - b - a}{b - a} = \frac{2x - 1 - 0}{1 - 0} = 2x - 1 \qquad x = \frac{z+1}{2} \qquad F(z) = e^{-(z+1)} \quad -1 \geq z \geq +1$$

The expansion becomes: $F(z) = \exp[-(z + 1)] \approx P_1(z) = A_0 T_0(z) + A_1 T_1(z)$

where $T_0(x) = 1$ and $T_1(x) = x$

The exact values of these coefficients are given by the integrals:

$$A_n = \frac{\displaystyle\int_{-1}^{+1} \frac{1}{\sqrt{1-z^2}}\; F(z)\; T_n(z)\; dz}{\displaystyle\int_{-1}^{+1} \frac{1}{\sqrt{1-z^2}}\; \left[T_n(z)\right]^2\; dz} \quad , \quad n = 0, 1, 2, \ldots$$

Substituting the given function:

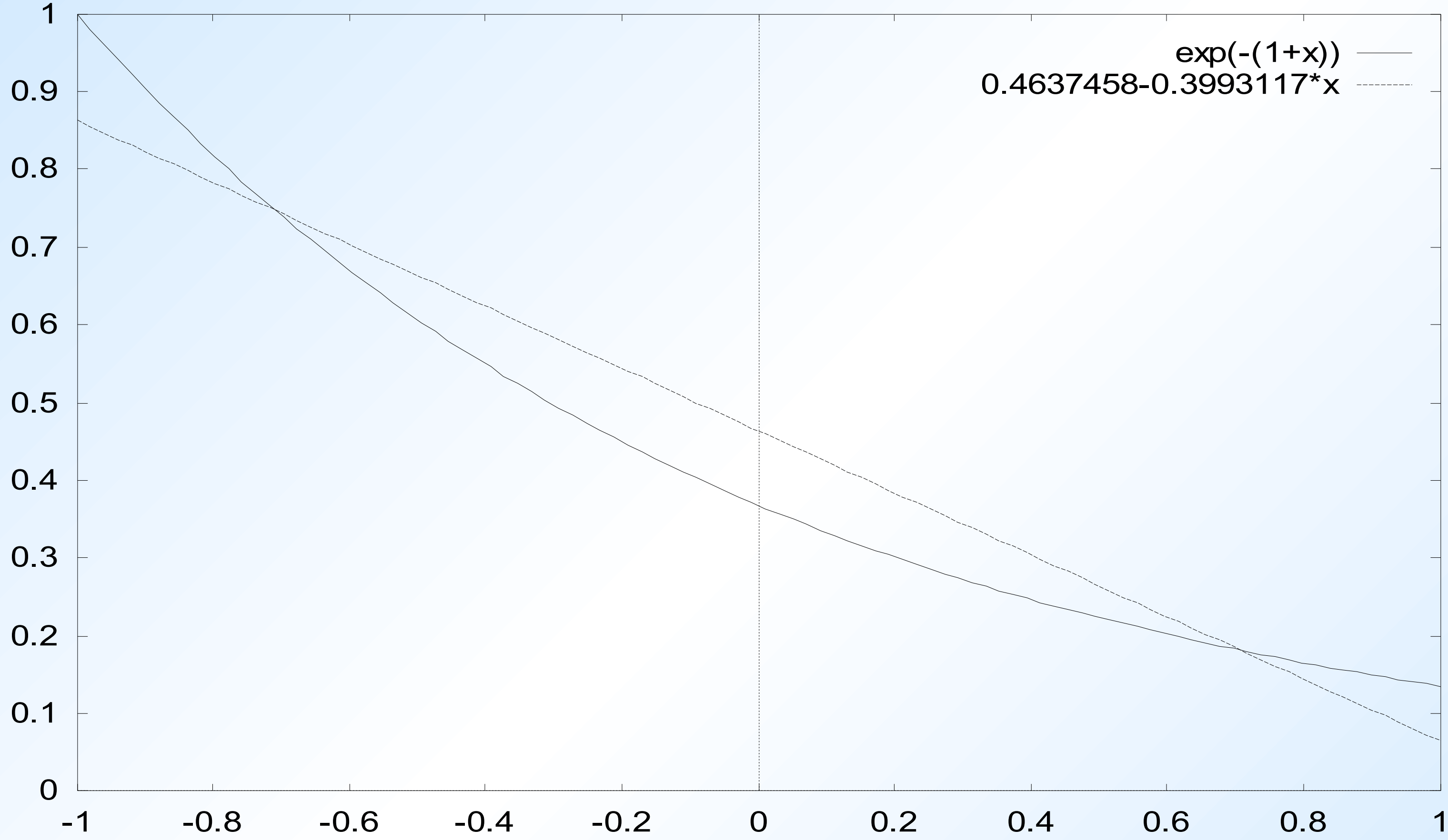
$$A_0 = \frac{1}{\pi} \int_{-1}^{+1} \frac{1}{\sqrt{1-z^2}}\; e^{-(z+1)}\; dz$$

$$A_1 = \frac{1}{\pi} \int_{-1}^{+1} \frac{z}{\sqrt{1-z^2}}\; e^{-(z+1)}\; dz$$

One can use a numerical integration method to evaluate the integrals and find these coefficients. Use one of the accurate methods and obtain

$A_0 = 0.4637458$	$A_1 = -\,0.3993117$
The polynomial is :	$P_1(z) = 0.4637458 - 0.3993117\,z$

The plot is given below



It can be observed that the cross-over points are symmetrical with respect to $z = 0$, and they are at $z = \pm 0.7071068 = 1/\sqrt{2}$

Solution 2: Using Chebyshev Economization

The second way is to use the MacLaurin series expansion of F(z):

$$F(z) = e^{-(z+1)} = e^{-1}\,(1 - z + \frac{z^2}{2} - \frac{z^3}{6} + \frac{z^4}{24} - \ldots)$$

Using reciprocal relations, change z 's to T 's:

$$F(z) = e^{-(z+1)} = e^{-1}\left(T_0 - T_1 + \frac{1}{2}\left(\frac{T_0 + T_2}{2}\right) - \frac{1}{6}\left(\frac{3\,T_1 + T_3}{4}\right) + \frac{1}{24}\left(\frac{3\,T_0 + 4\,T_2 + T_4}{8}\right) - \ldots\right)$$

$$F(z) = e^{-1}\left(\frac{81}{64}T_0 - \frac{9}{8}T_1 + \frac{13}{48}T_2 - \frac{1}{24}T_3 + \frac{1}{192}T_4 - \ldots\right)$$

Economize, i.e., drop high-degree terms including T_2 and change back to z ' s.

$$P_1(z) \approx e^{-1}\left(\frac{81}{64} - \frac{9}{8}z\right) = 0.4655974 - 0.413644\,z$$

This is approximately the required first-degree polynomial. It is approximate because the coefficients are determined approximately when only the first five terms in the MacLaurin expansion are used. For a more accurate answer, one needs to use more number of terms.

Compare the these coefficients with those of the previous solution.

Solution 3:

Use the property that the required $P_1(z)$ crosses $F(z)$ exactly at the roots of $T_2(z) = 0$.

The roots of $T_2(z) = 2z^2 - 1 = 0$ are $z_1 = -1/\sqrt{2}$ and $z_2 = 1/\sqrt{2}$

At $z_1 = -1/\sqrt{2}$ $F(z_1) = \exp(1 - 1/\sqrt{2})$

At $z_2 = 1/\sqrt{2}$ $F(z_2) = \exp(1 + 1/\sqrt{2})$

The first-degree polynomial that passes through these points is:

$P_1(z) = 0.4637458 - 0.3993117\, z$

This is exactly the same polynomial found in the first solution procedure.