Basic Classification of PDE's

$$\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{i,j} \frac{\partial u}{\partial x_j} \right) + \sum_{i=1}^{d} b_i \frac{\partial u}{\partial x_i} + c\,u = f$$

Classification according to

- Order (Highest order of the partial derivate in the PDE)
- Number of independent variables (x, y, z, t, ...)
- Linearity [of the dependent variable, u(x,y,...) and all the derivatives of u]
- Constant / variable coefficients (of u and all the derivatives of u)
- Homogenity (?)

Mathematical Classification of Second-order PDE's

Consider the second order equation: $$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} = f$$

$$a\, u_{xx} + b\, u_{xy} + c\, u_{yy} = f\,(x, y, u, u_x, u_y)$$

where a, b, c, and f are all functions of x, y, u, u_x, and u_y, in general. The directional derivatives of u_x and u_y must also exist, i.e.:

$$d\,(u_x) = u_{xx}\, d\,x + u_{xy}\, d\,y$$

$$d\,(u_y) = u_{xy}\, d\,x + u_{yy}\, d\,y$$

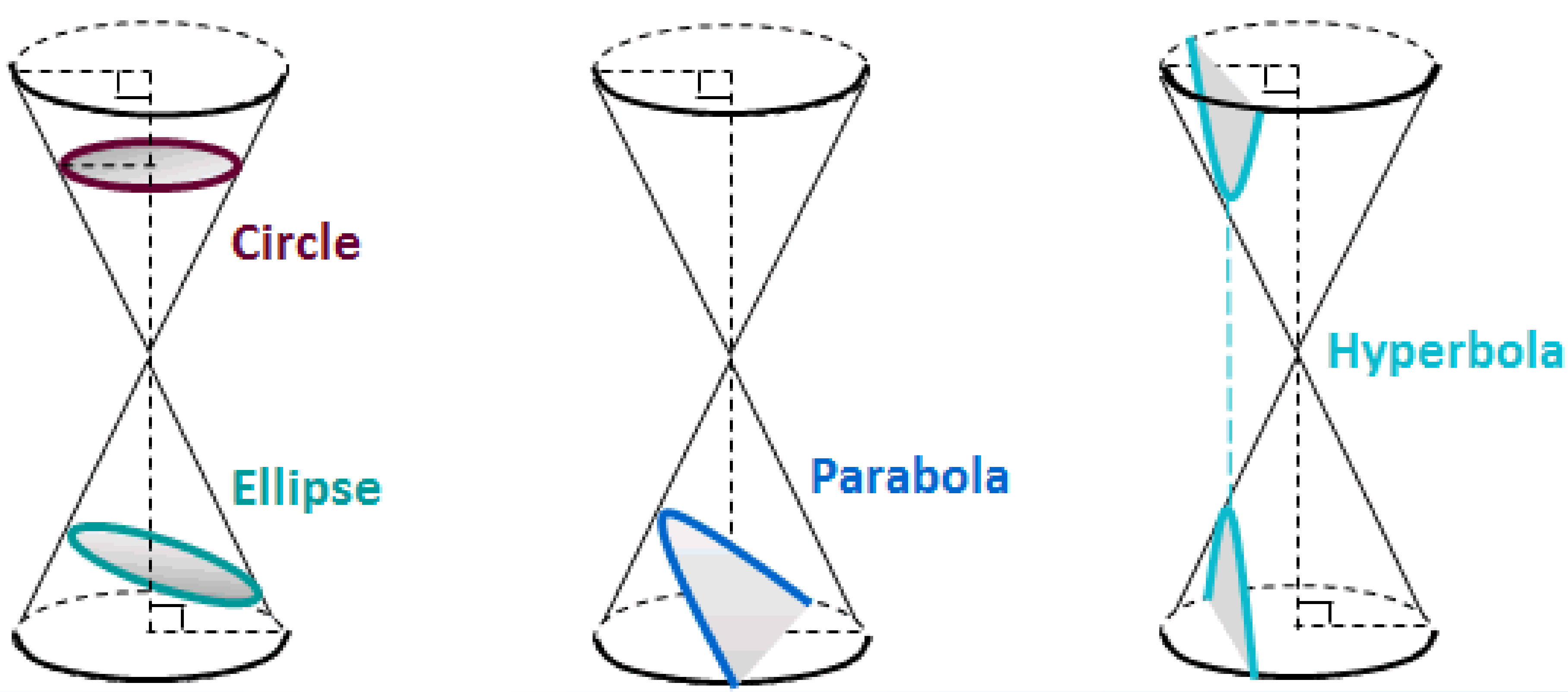
In matrix notation: $$\begin{pmatrix} a & b & c \\ dx & dy & 0 \\ 0 & dx & dy \end{pmatrix} \begin{pmatrix} u_{xx} \\ u_{xy} \\ u_{yy} \end{pmatrix} = \begin{pmatrix} f \\ d(u_x) \\ d(u_y) \end{pmatrix}$$

The solution for u_{xx}, u_{xy}, and u_{yy} exists, and it is unique unless the determinant of the coefficients matrix vanishes:

$$a\,(d\,y)^2 - b\,(d\,y)\,(d\,x) + c\,(d\,x)^2 = 0$$

This is called the characteristic equation.

If	$b^2 - 4\,a\,c > 0$	Hyperbolic PDE
	$b^2 - 4\,a\,c = 0$	Parabolic PDE
	$b^2 - 4\,a\,c < 0$	Elliptic PDE


Circle
Ellipse
Parabola
Hyperbola

Examples:

Transient heat conduction equation (parabolic):

"Propagation"

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{\alpha} \frac{\partial U}{\partial t}$$

Steady heat conduction equation (elliptic):

"Equilibrium"

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$$

Compressible fluid flow or

Vibration of a string (hyperbolic)

"Eigenvalue"

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{c_1^2} \frac{\partial^2 U}{\partial t^2}$$

For the solution of a hyperbolic PDE, you may use all the methods we have discussed before on ellipic and parabolic types, with similar arguments.

However, the hyperbolic type PDE, has another type of solution, quite different from the finite-differences we used before.

This situation is somewhat similiar to the method-of lines used for parabolic PDE's. The method-of lines is half finite differences.

This different type of solution is called the method of characteristics.

The method of characteristics is specific to first-order PDE's and second-order hyperbolic PDE's, but not second-order parabolics and elliptics.

Let's first see the solution of first-order PDE's.

First-order PDE's and the Method of Characteristics

Consider the transport equation $$a\frac{\partial u}{\partial t} + b\frac{\partial u}{\partial x} = f$$

u (say, concentration) depends on t (time) and x (space) with some initial condition u(0,x) and one boundary condition. Suppose that a, b, and f are constants, for simplicity.

Problem: Find u(t,x) for t > 0 and any x where t and x are independent parameters.

Change the PDE into an ODE! How?

Come up with a relationship between the independent parameters, t and x, by finding a common parameter, s, such that x = x(s) and t = t(s) so that u = u(s). Then, PDE with t and x becomes ODE with s? Possible?

This is the **method of characteristics** in a nut shell.

Suppose there is such a parameter, s. Then $\dfrac{du}{ds} = \dfrac{\partial u}{\partial t}\dfrac{dt}{ds} + \dfrac{\partial u}{\partial x}\dfrac{dx}{ds}$

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt}$$

Given PDE $a\dfrac{\partial u}{\partial t} + b\dfrac{\partial u}{\partial x} = f$ => $\dfrac{f}{a} = \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x}\dfrac{b}{a}$

If $\dfrac{dx}{dt} = \dfrac{b}{a}$ Then $\dfrac{du}{dt} = \dfrac{f}{a}$ Integrate both (assuming constants)

If $\underbrace{x(t) = x_0 + \dfrac{b}{a}t}_{\text{Characteristic Equation}}$ Then $\underbrace{u(t) = u_0 + \dfrac{f}{a}t}_{\text{Solution Equation}}$ What if they are not constants?

Characteristic Equation **Solution Equation**

Example: Transport Equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \quad , \quad u(0,x) = u_0(x)$$

c is the constant speed of the wave

Characteristic equation: $\frac{dx}{dt} = c \quad \Rightarrow \quad \boxed{x(t) = c\,t + x_0}$

Characteristic curves are straight lines in the *x* - t plane with slope 1/*c*

Solution equation: $\frac{du}{dt} = 0 \quad \Rightarrow \quad \boxed{u(t) = u_0}$

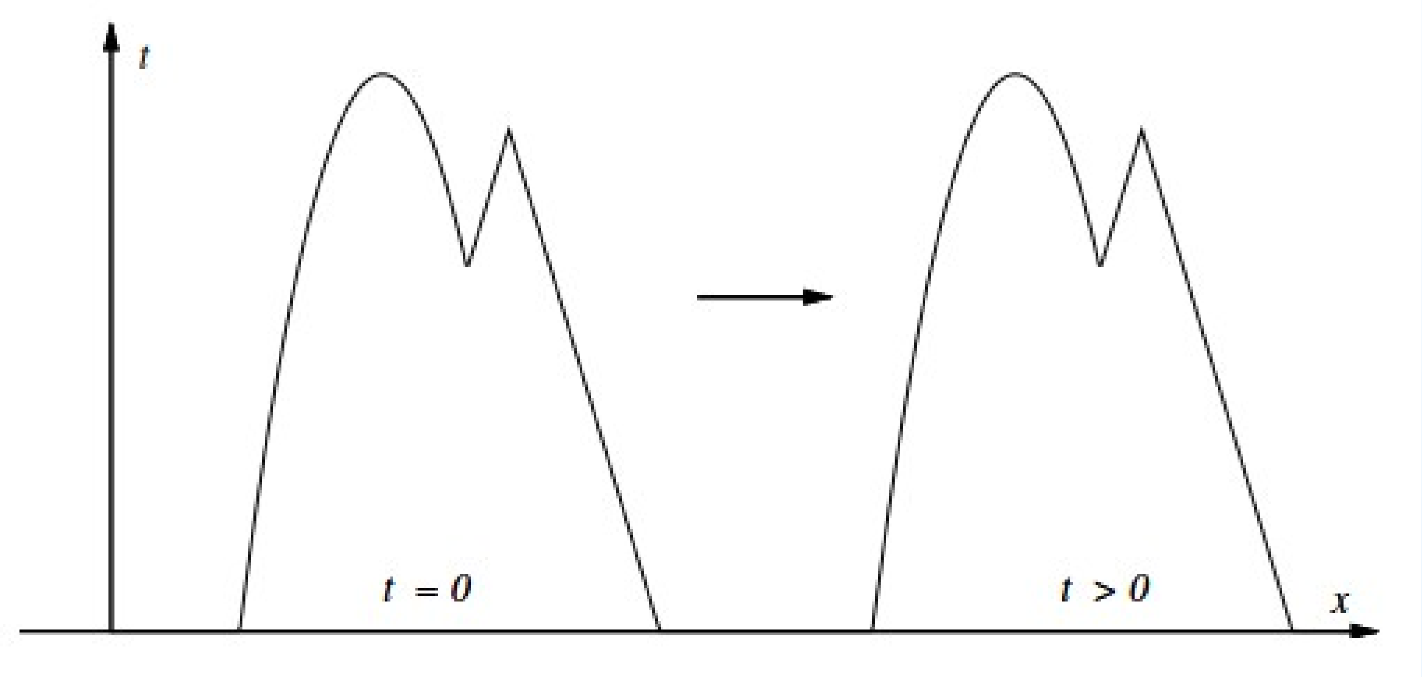
The parameter that relates x with t is $s = x - c\,t$

$$u(s) = \boxed{u(t,x) = u_0\left(x - c\,t\right)}$$

The solution u is constant along the chacteristic lines.

It represents a wave traveling to the right at speed *c*, maintainin its initial shape.

The solution can be regarded as a wave that propagates with speed a without change of shape, as illustrated



The book chapter on solution of first-order PDE’s is on OdtuClass.

Example: Transport Equation – Finite Difference Solution

$$\frac{\partial u}{\partial t} + c\,\frac{\partial u}{\partial x} = 0 \quad , \quad u(0,x) = u_0(x)$$

c is the constant speed of the wave

Replace derivatives with finite differences. Here are few examples:

$$\frac{u_{i,j+1} - u_{i,j}}{k} + c\left(\frac{u_{i+1,j} - u_{i,j}}{h}\right) = 0 \qquad \frac{u_{i,j+1} - u_{i,j}}{k} + c\left(\frac{u_{i,j} - u_{i-1,j}}{h}\right) = 0$$

$$\frac{u_{i,j+1} - u_{i,j}}{k} + c\left(\frac{u_{i+1,j} - u_{i-1,j}}{2\,h}\right) = 0 \qquad \frac{u_{i,j+1} - u_{i,j-1}}{2\,k} + c\left(\frac{u_{i+1,j} - u_{i-1,j}}{2\,h}\right) = 0$$

Leapfrog scheme

$$\frac{u_{i,j+1} - \frac{1}{2}\left(u_{i+1,j} + u_{i-1,j}\right)}{k} + c\left(\frac{u_{i+1,j} - u_{i-1,j}}{2\,h}\right) = 0$$

Lax–Friedrichs scheme

The Lax–Richtmyer Equivalence Theorem: A consistent finite difference scheme for a partial differential equation for which the initial value problem is well-posed is convergent if and only if it is stable.

There are no explicit, unconditionally stable, consistent finite difference schemes for hyperbolic systems of partial differential equations

Stability condition for explicit schemes: $$r\,c \leq 1 \quad , \quad r = \frac{k}{h}$$

Courant–Friedrichs–Lewy condition

Some implicit schemes are stable for all r = k/h.

Second-order Hyperbolic PDE's

Many of the problems encountered in science and engineering are of second order and take the general quasi-linear form

$$a \frac{\partial^2 u}{\partial x^2} + b \frac{\partial^2 u}{\partial x \, \partial y} + c \frac{\partial^2 u}{\partial y^2} = e \quad \text{............} \quad (5)$$

where a, b, c, and e are functions of x, y, u_x, and u_y.

For a solution to be possible, we again require some initial condition specified.

The analysis of the first-order PDE can be extended to second-order equations.

That is, along some curve τ in the x-y plane, we require that a parameter s exists such that

$$x = x(s) \quad \text{and} \quad y = y(s)$$

$$\frac{du}{ds} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds} \quad \text{.............} \quad (6)$$

$$\frac{d}{ds}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2}\frac{dx}{ds} + \frac{\partial^2 u}{\partial y\,\partial x}\frac{dy}{ds} \quad \text{.............} \quad (7)$$

$$\frac{d}{ds}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial y^2}\frac{dy}{ds} + \frac{\partial^2 u}{\partial x\,\partial y}\frac{dx}{ds} \quad \text{.............} \quad (8)$$

Eliminate $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$ from (5), (7), and (8), the PDE becomes

$$\left[a\left(\frac{dy}{ds}\right)^2 - b\left(\frac{dx}{ds}\frac{dy}{ds}\right) + c\left(\frac{dx}{ds}\right)^2\right]\frac{\partial^2 u}{\partial x\,\partial y} =$$

$$a\frac{dy}{ds}\frac{d}{ds}\left(\frac{\partial u}{\partial x}\right) - e\frac{dy}{ds}\frac{dx}{ds} + c\frac{dx}{ds}\frac{d}{ds}\left(\frac{\partial u}{\partial y}\right) \quad \text{.............} \quad (9)$$

We can eliminate the effect of $\dfrac{\partial^2 u}{\partial x\, \partial y}$ by defining

$$a\left(\frac{dy}{ds}\right)^2 - b\left(\frac{dx}{ds}\,\frac{dy}{ds}\right) + c\left(\frac{dx}{ds}\right)^2 = 0$$

Unless $\dfrac{dx}{ds} = 0$, this can be written as a quadratic equation in $\dfrac{dy}{dx}$

$$a\left(\frac{dy}{dx}\right)^2 - b\left(\frac{dy}{dx}\right) + c = 0 \quad \text{.............} \quad (10)$$

Characteristic Eq'n.

Equation (9) becomes:

$$e\,dy - a\,\frac{dy}{dx}\,d\left(\frac{\partial u}{\partial x}\right) - c\,d\left(\frac{\partial u}{\partial y}\right) = 0 \quad \text{.......} \quad (11)$$

Solution Eq'n.

For a point (x,y) associated with the given values of u and $\partial u / \partial x$, there will be two directions for which the characteristic equation (10) is satisfied provided that

$$b^2 > 4\,a\,c$$

which is the definition of a hyperbolic PDE.

If $b^2 = 4\,a\,c$ (parabolic PDE), the characteristics are coincident.

If $b^2 < 4\,a\,c$ (elliptic PDE), we have complex roots.

Parabolic and elliptic PDE's, therefore, cannot be solved with the method of characteristics. However, all can be solved by Taylor series expansion and finite differences.

Let us simplify the above notation by defining $p = \frac{\partial u}{\partial x}$ and $q = \frac{\partial u}{\partial y}$

Equation (6) becomes: $du = p\,dx + q\,dy$ .... (12)

Equation (11) becomes:

Solution Equation

$$e\,dy - a\frac{dy}{dx}\,dp - c\,dq = 0 \quad .... \quad (13)$$

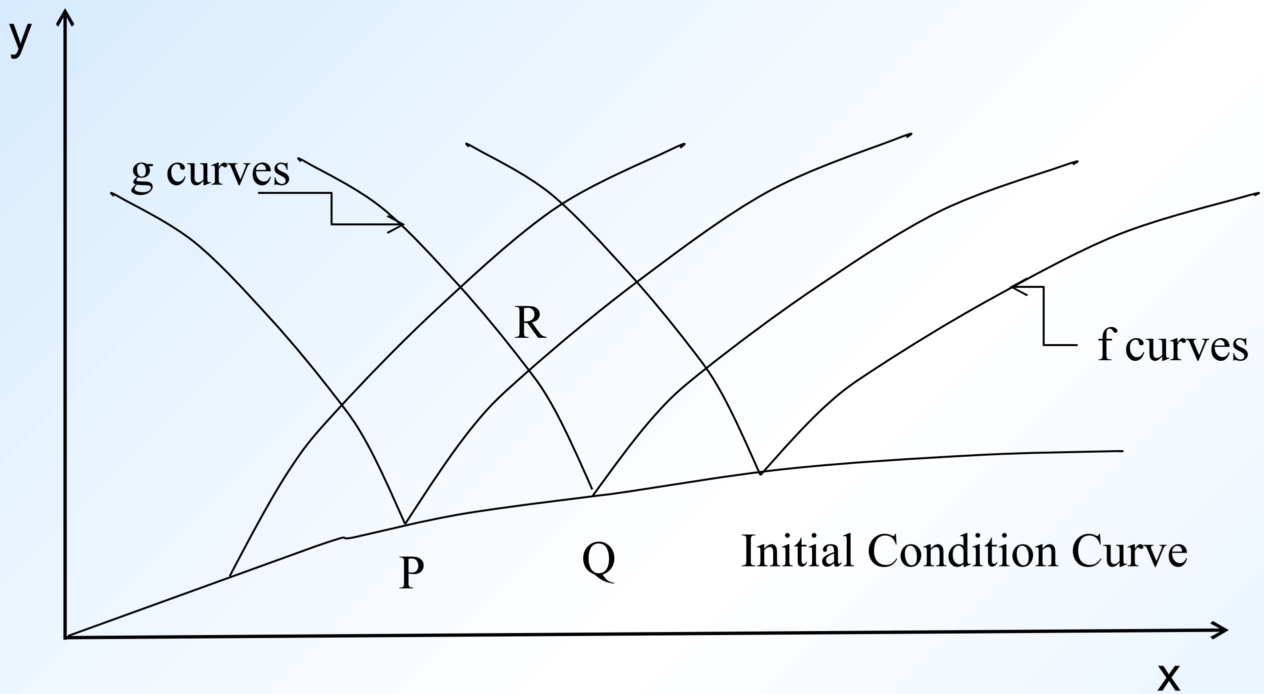
Characteristic Eq'n.

$$a\left(\frac{dy}{dx}\right)^2 - b\left(\frac{dy}{dx}\right) + c = 0 \quad .... \quad (10)$$

Characteristic equation yields two real solutions for dy/dx. Define:

$$\frac{dy}{dx} = f(x, y, u, p, q) \quad \text{and} \quad \frac{dy}{dx} = g(x, y, u, p, q)$$

f and g curves are the characteristic curves



We can obtain a solution at R, by integrating both the characteristic equation and the solution equation from P to R along f curve, and from Q to R along g curve.