Numerical Integration

Given a complicated f(x) or a set of tabulated values, x_i's and corresponding $f(x_i)$'s:

Question: For a given range, [a,b]

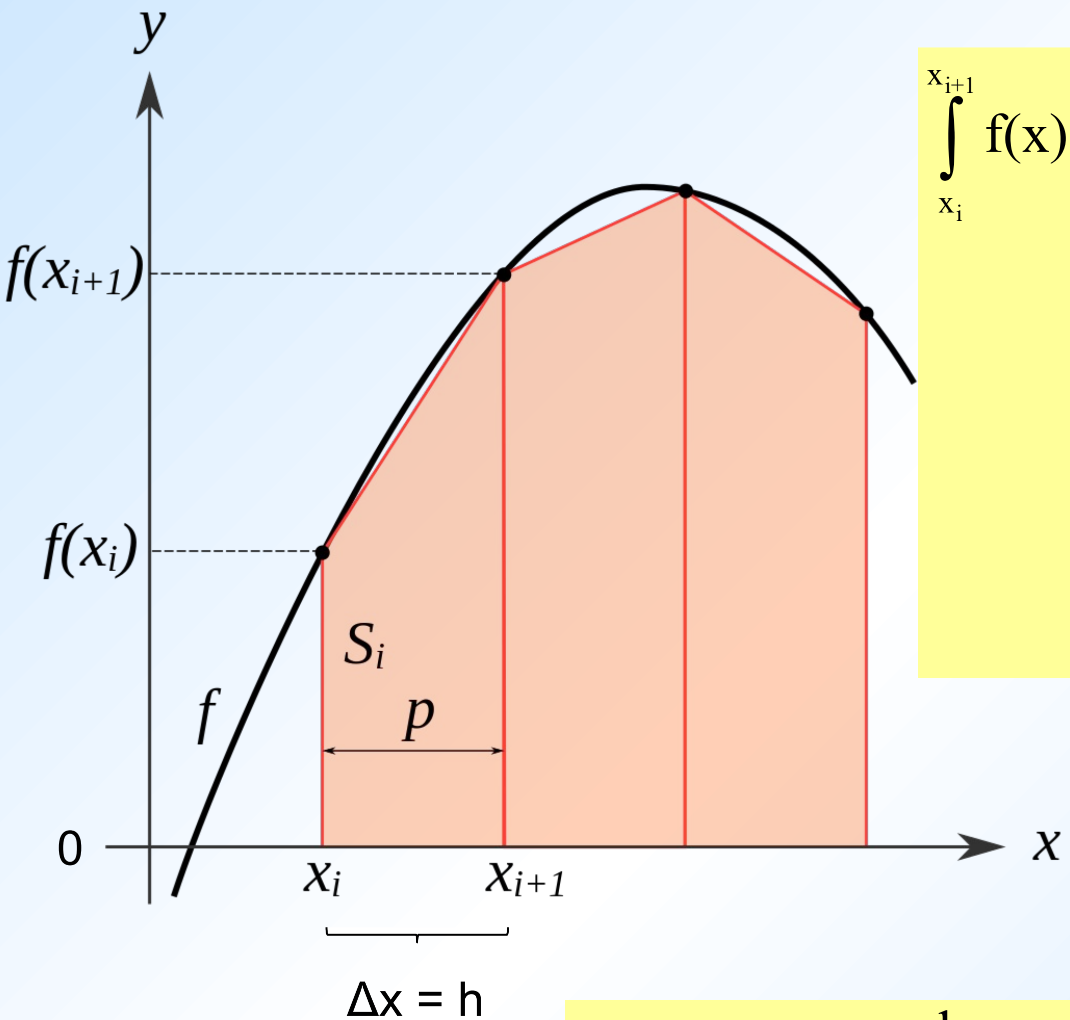
$$\int_a^b f(x)\,dx = ?$$

Answer: Replace f (x) or the tabulated data with a simple function g(x) and operate on g(x), instead

$$f(x) \cong g(x) \quad , \quad \int_a^b f(x)\,dx \cong \int_a^b g(x)\,dx$$

Error of integration:

$$E = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$



$$\int_{x_i}^{x_{i+1}} f(x)\,dx \cong \int_{x_i}^{x_{i+1}} P_1(x)\,dx$$

$$\cong h\,f(x_i) + \frac{h}{2}\left[f(x_{i+1}) - f(x_i)\right]$$

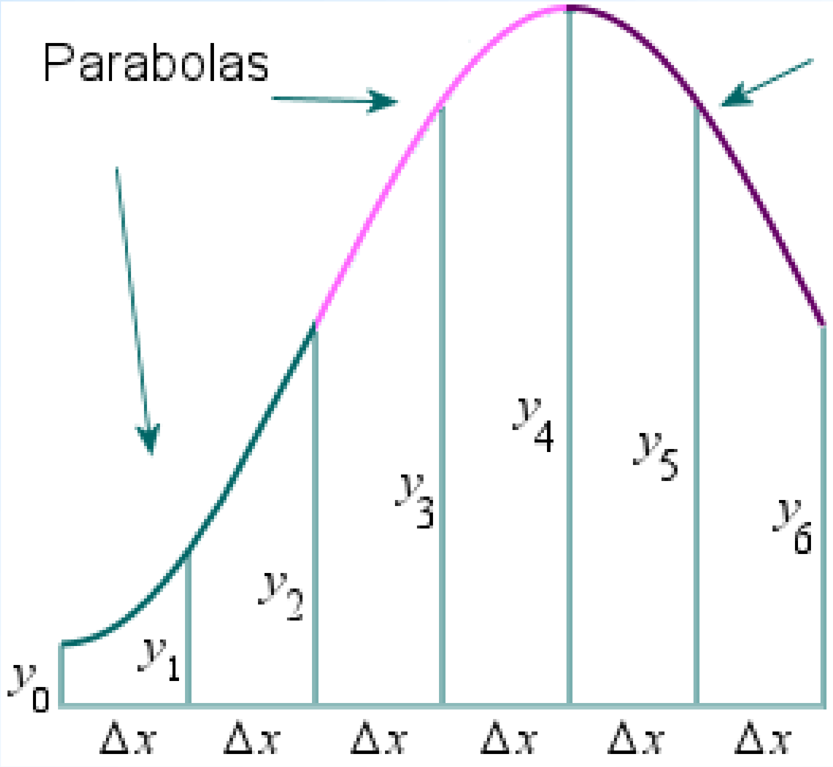
$$\cong \frac{h}{2}\left[f(x_i) + f(x_{i+1})\right]$$

$$\cong \frac{h}{2}\left[f_i + f_{i+1}\right]$$

Trapezoidal Rule

$$\int_{x_0}^{x_n} f(x)\,dx \cong \frac{h}{2}\left(f_0 + 2f_1 + 2f_2 + \dots + 2f_{n-1} + f_n\right)$$

Second-degree Polynomials, $P_2(x)$



$$\int_{x_0}^{x_2} f(x)\,dx \cong \int_{x_0}^{x_2} P_2(x)\,dx$$

$$P_2(x) = f_0 + s\,\Delta f_0 + \frac{s\,(s-1)}{2\,!}\,\Delta^2 f_0$$

$$s = \frac{x - x_0}{h}$$

Simpson's One-third Rule

4's and 2's alternate

$$\int_{x_0}^{x_n} f(x)\,dx \cong \frac{h}{3}\left(f_0 + 4 f_1 + 2 f_2 + \ldots + 2 f_{n-2} + 4 f_{n-1} + f_n\right)$$

Newton - Cotes Formulae

Trapezoidal Rule (set of first-degree polynomials)

$$\int_{x_0}^{x_n} f(x)\, dx \cong \frac{h}{2}\left(f_0 + 2 f_1 + 2 f_2 + \ldots + 2 f_{n-1} + f_n\right)$$

$$E_{trap} = \frac{-(x_n - x_0)^3}{12\, n^2} f''(\xi)$$

number of panels

Simpson's One-third Rule (set of second-degree polynomials)

$$\int_{x_0}^{x_n} f(x)\, dx \cong \frac{h}{3}\left(f_0 + 4 f_1 + 2 f_2 + \ldots + 2 f_{n-2} + 4 f_{n-1} + f_n\right)$$

$$E_{1/3} = \frac{-(x_n - x_0)^5}{180\, n^4} f^{(4)}(\xi)$$

Simpson's Three-eights Rule (set of third-degree polynomials)

$$\int_{x_0}^{x_n} f(x)\, dx \cong \frac{3h}{8}\left(f_0 + 3 f_1 + 3 f_2 + 2 f_3 + \ldots + 2 f_{n-3} + 3 f_{n-2} + 3 f_{n-1} + f_n\right)$$

Thomas Simpson FRS

British Mathematician

1710 - 1761

Newton-Cotes Formulae:

$$I = \int_a^b f(x)\, dx \cong \sum_{i=0}^{n} \underbrace{w_i}_{\text{weights}} f(\underbrace{x_i}_{\text{nodes}})$$

The range, (a,b), is divided into equal spacings (panels).

A polynomial of small degree passes through these equally-spaced points.

Why equal spacing?

Does the function has to be a polynomial?

Error in Newton-Cotes Formulae:

$$\underbrace{I = \int_a^b f(x)\, dx}_{\text{Exact}} = \underbrace{\sum_{i=1}^{N} w_i\, f(x_i)}_{\text{Approx.}} + \int_a^b g(x)\, h(x)\, dx$$

How to minimize?

Infinite Set of Orthogonal Functions:

Two functions, f(x) and g(x), both real-valued and continous, are called orthogonal to each other with respect to a third (weighting) function, w(x), in the range [a,b] if

$$I = \int_a^b w(x)\, f(x)\, g(x)\, dx = 0$$

Examples:

$$\int_0^1 (1)\sin(n\pi x)\sin(m\pi x)\, dx = 0 \quad \text{when } n \neq m$$

$$\{\sin(n\pi x)\ ,\ n = 1, 2, \ldots \infty\}$$

$$\int_0^1 (1)\cos(n\pi x)\cos(m\pi x)\, dx = 0 \quad \text{when } n \neq m$$

$$\{\cos(n\pi x)\ ,\ n = 0, 1, \ldots \infty\}$$

Fourier Series

Jean-Baptiste Joseph Fourier

French Mathematician

1768 -1830

Legendre polynomials, which form an infinite set, have this property when w(x) = 1 and the range [-1,+1]

$$\int_{-1}^{1} (1)\, P_n(x)\, P_m(x)\, dx = 0 \quad , \quad n \neq m$$

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

$$\{P_n(x)\ ,\ n = 0, 1, \ldots \infty\}$$

Adrien-Marie Legendre

French Mathematician

1752 – 1833

Legendre Polynomials

$P_0(x) = 1$

$P_1(x) = x$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

Chebyshev Polynomials

$T_0(x) = 1$	$T_1(x) = x$	$T_2(x) = 2x^2 - 1$
$T_3(x) = 4x^3 - 3x$	$T_4(x) = 8x^4 - 8x^2 + 1$	$T_5(x) = 16x^5 - 20x^3 + 5x$

Laguerre Polynomials

$$L_0^0(x) = 1$$

$$L_1^0(x) = -x + 1$$

$$L_2^0(x) = \frac{1}{2}(x^2 - 4x + 2)$$

$$L_3^0(x) = \frac{1}{6}(-x^3 + 9x^2 - 18x + 6)$$

$$L_4^0(x) = \frac{1}{24}(x^4 - 16x^3 + 72x^2 - 96x + 24)$$

Hermite Polynomials

$H_0(x) = 1$	$H_1(x) = 2x$	$H_2(x) = 4x^2 - 2$
$H_3(x) = 8x^3 - 12x$	$H_4(x) = 16x^4 - 48x^2 + 12$	$H_5(x) = 32x^5 - 160x^3 + 120x$

ORTHOGONAL POLYNOMIALS

Name	Range of Orthogonality	Weighting Function	Integral Property
Legendre	[- 1 , + 1]	w(x) = 1	$\int_{-1}^{+1} w\, P_k\, P_k\, dx = \frac{2}{2k+1}$
Chebyshev	[- 1 , + 1]	$w(x) = \frac{1}{\sqrt{1-x^2}}$	$\int_{-1}^{+1} w\, T_k\, T_k\, dx = \pi \quad \text{if } k = 0$ $\int_{-1}^{+1} w\, T_k\, T_k\, dx = \frac{\pi}{2} \quad \text{if } k > 0$
Laguerre	[0 , ∞]	$w(x) = x^{\alpha} e^{-x}$	$\int_{0}^{\infty} w\, L_k^{\alpha}\, L_k^{\alpha}\, dx = \frac{\Gamma(\alpha+k+1)}{k!}$
Hermite	[- ∞ , +∞]	$w(x) = e^{-x^2}$	$\int_{-\infty}^{+\infty} w\, H_k\, H_k\, dx = \sqrt{\pi}\; 2^k\; k!$

Admond Nicolas Laguerre

French Mathematician

1834 – 1886

Expansion of f(x) in terms of infinite set of orthogonal functions

Suppose the functions, $Q_n(x)$, forms an orthogonal set, n -> inf.

$$f(x) = c_0\, Q_0(x) + c_1\, Q_1(x) + c_2\, Q_2(x) + \dots + c_n\, Q_n(x) + \dots$$

$$\int_a^b w(x)\, Q_n(x)\, f(x)\, dx =$$

$$\int_a^b w(x)\, Q_n(x)\left(c_0 \cancel{Q_0(x)} + c_1 \cancel{Q_1(x)} + c_2 \cancel{Q_2(x)} + \dots + c_n\, Q_n(x) + \cancel{\dots}\right) dx$$

$$= \int_a^b w(x)\, Q_n(x)\left(c_n\, Q_n(x)\right) dx$$

$$c_n = \frac{\int_a^b w(x)\, Q_n(x)\, f(x)\, dx}{\int_a^b w(x)\, Q_n^2(x)\, dx}$$

Example

Expand the function f(x) = 1 in Fourier sine series in the range [0,1], i.e., find the coefficients, a_n's, in the orthogonal expansion

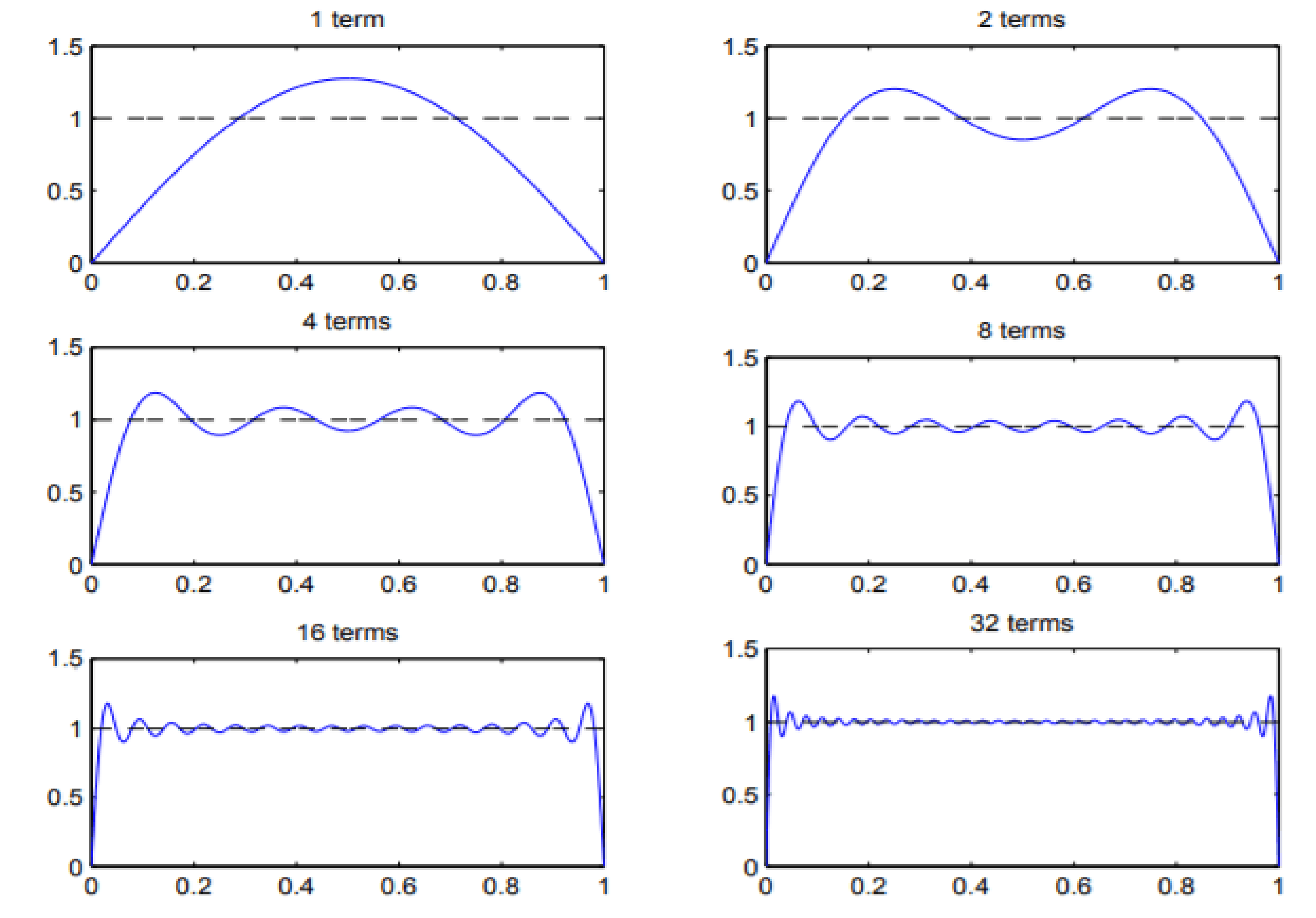
$$f(x) = \sum_{0}^{\infty} a_n \sin(n\pi x)$$

The orthogonality property: $$\int_0^L \sin\left(\frac{m\,\pi\,x}{L}\right)\sin\left(\frac{n\,\pi\,x}{L}\right)dx = \begin{cases} L/2 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

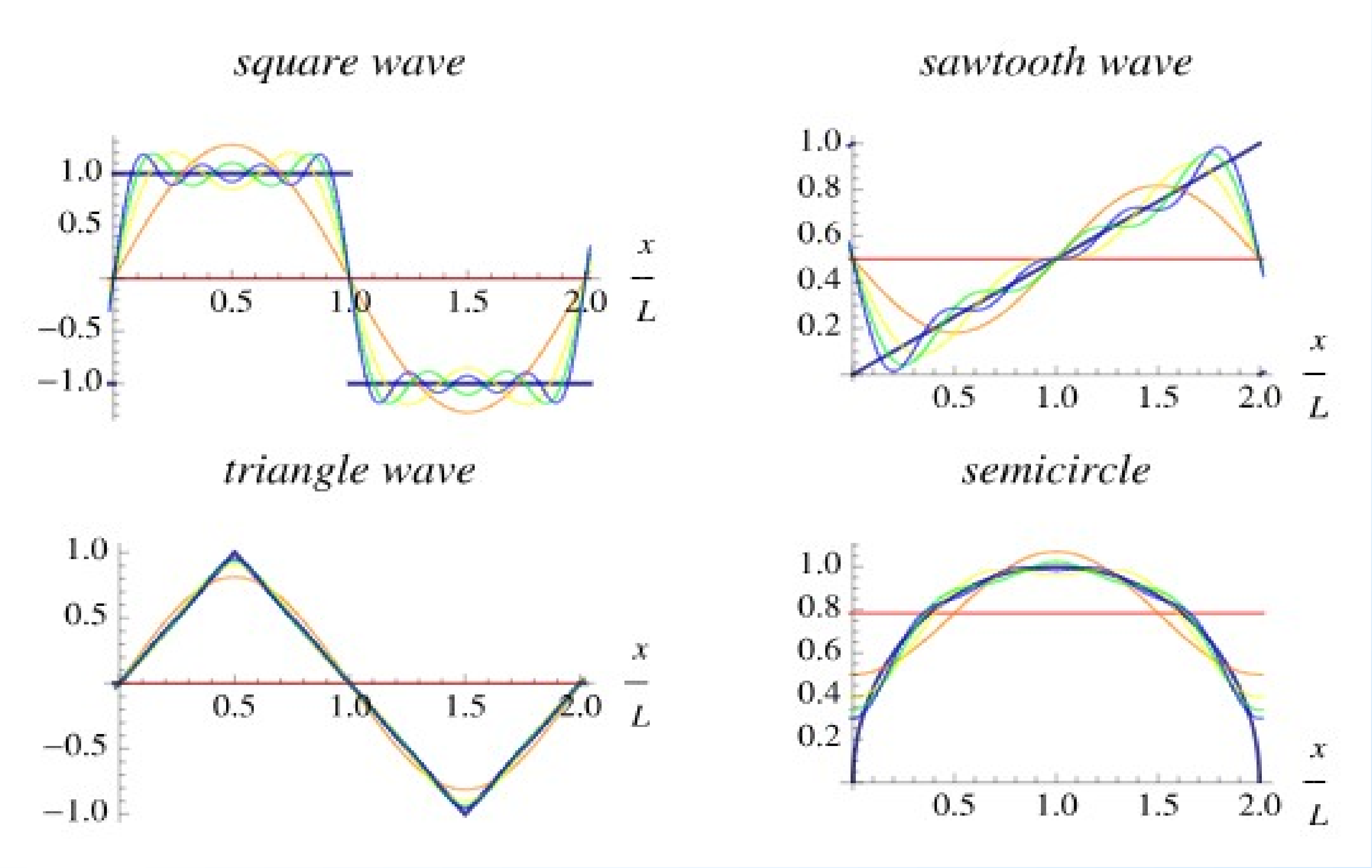
$$\int_0^1 \sin(n\pi x)\,dx = -\frac{1}{n\,\pi}\left[\cos(n\pi x)\Big|_0^1\right] = \frac{2}{n\,\pi} \quad \text{if } n \text{ is odd}$$

$$= 0 \quad \text{if n is even}$$

Solution: $$f(x) = 1 = \frac{4}{\pi}\sin(\pi x) + \frac{4}{3\pi}\sin(2\pi x) + \frac{4}{5\pi}\sin(3\pi x) + ...$$



A Fourier series is an expansion of a periodic function, F(x), in terms of an infinite sum of sines and cosines.



Expressing a periodic function, F(x), as a Fourier series (sines and cosines) is sometimes more advantageous than expanding it as a power series (Taylor or MacLaurin). In particular, astronomical phenomena are usually periodic, as are heartbeats, tides, and vibrating strings, so it makes sense to express them in terms of periodic functions.

Fourier series make use of the orthogonality relationships of the sine and cosine functions. The computation and study of Fourier series is known as harmonic analysis and is extremely useful as a way to break up an *arbitrary* periodic function into a set of simple terms that can be plugged in, solved individually, and then recombined to obtain the solution to the original problem or an approximation to it to whatever accuracy is desired or practical.

The original motivation of Fourier was to solve the heat equation which is a parabolic partial differential equation:

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+q_V=\rho c_p\frac{\partial T}{\partial t}$$

See the book chapter, «Fourier Method for Heat Conduction», on OdtuClass.

Orthogonal fuctions as polynomials (such as Legendre set), not as periodic functions (such as sines and cosines), have another important use: evaluate integrals very accurately. The original work was done by Gauss.

Carl Friedrich Gauss

German Mathematician

1777 – 1855

ORTHOGONAL POLYNOMIALS

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Laguerre	[0 , ∞]	$w(x) = x^{\alpha} e^{-x}$	$\int_{0}^{\infty} w\, L_k^{\alpha}\, L_k^{\alpha}\, dx = \frac{\Gamma(\alpha+k+1)}{k!}$
Hermite	[- ∞ , +∞]	$w(x) = e^{-x^2}$	$\int_{-\infty}^{+\infty} w\, H_k\, H_k\, dx = \sqrt{\pi}\, 2^k\, k!$

Gauss - Legendre Quadrature Method

$$I = \int_a^b f(x)\, dx \cong \sum_{i=0}^{n} w_i\, f(x_i)$$

n + 1 values of w_i

n + 1 values of $f(x_i)$

=> *There are 2 (n + 1) parameters which define a polynomial of degree 2 n + 1*

Choose x_i's such that, the sum $\sum_{i=0}^{n} w_i\, f(x_i)$ gives the integral

$\int_a^b f(x)\, dx$ exactly when f (x) is a polynomial of degree 2 n + 1, or less

Replace f(x) by some polynomial of degree n

$$I = \int_a^b f(x)\, dx = \int_a^b P_n(x)\, dx + \int_a^b R(x)\, dx$$

where the last term is the error term.

We will try to make $$\int_a^b R(x)\, dx = 0$$

when f (x) is a polynomial of degree 2 n + 1, or less.

Write $P_n(x)$ in Lagrange form:

$$P_n(x) = \sum_{i=0}^{n} L_i(x)\; f(x_i)$$

$$P_n(x) = \underbrace{\frac{(x - x_1)(x - x_2)\ldots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)}}_{L_0(x)} f(x_0) + \underbrace{\frac{(x - x_0)(x - x_2)\ldots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_n)}}_{L_1(x)} f(x_1) + \;.....$$

Substitute

$$I = \int_a^b f(x)\; dx = \int_a^b \sum_{i=0}^{n} L_i(x)\; f(x_i)\; dx + \int_a^b R(x)\; dx$$

$$\cong \sum_{i=0}^{n} \left[\int_a^b L_i(x)\, dx \right] f(x_i) + \int_a^b R(x)\; dx$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\sum_{i=0}^{n} w_i\; f(x_i) \qquad\qquad = 0 \quad ??$$

The error term in Lagrange form:

$$R(x) = \prod_{i=0}^{n} (x - x_i) \frac{f^{(n+1)}(\xi)}{(n+1)!} \qquad , \qquad a \le \xi \le b$$

Change coordinates from x in [a , b] to z in [-1 , 1] where

$$z = \frac{2x - (a+b)}{b-a} \quad , \quad dz = \frac{2}{b-a}\, dx$$

$$F(z) = f(x) = f\left(\frac{(b-a)z + b + a}{2}\right)$$

$$f(x) = F(z) = \sum_{i=0}^{n} L_i(z)\, F(z_i) + \prod_{i=0}^{n} (z - z_i) \frac{F^{(n+1)}(\xi)}{(n+1)!} \qquad , \qquad -1 \le \xi \le +1$$

where $$L(z) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{z - z_j}{z_i - z_j}$$

$$I = \int_a^b f(x)\,dx = \int_{-1}^{1} F(z)\,dz = \sum_{i=0}^{n} \left[\int_{-1}^{1} L_i(z)\,dz \right] F(z_i) + \int_{-1}^{1} R(z)\,dz$$

$$R(z) = \prod_{i=0}^{n} (z - z_i) \frac{F^{(n+1)}(\xi)}{(n+1)!} \quad , \quad -1 \le \xi \le +1$$

$f(x)$ or $F(z)$ is a polynomial of degree $2n + 1$, then $F^{(n+1)}(x)$ is a polynomial of degree n, at most.

Let $$\frac{F^{(n+1)}(\xi(z))}{(n+1)!} = g_n(z)$$

$$I = \int_{-1}^{1} F(z)\,dz = \sum_{i=0}^{n} \left[\int_{-1}^{1} L_i(z)\,dz \right] F(z_i) + \int_{-1}^{1} \prod_{i=0}^{n} (z - z_i)\; g_n(z)\; dz$$

<u>Question:</u> $\int_{-1}^{1} \prod_{i=0}^{n} (z - z_i)\; g_n(z)\; dz = 0$?? when F (z) is a polynomial of degree 2 n + 1.

The answer is to choose z_i's such that this is true, using a complete set of orthogonal functions.

Choose <u>Legendre Polynomials</u> as the orthogonal set of functions. Expand in terms of Legendre polynomials:

$$\prod_{i=0}^{n} (z - z_i) = b_0\, P_0\,(z) + b_1\, P_1\,(z) + \ldots + b_{n+1}\, P_{n+1}\,(z) = \sum_{i=0}^{n+1} b_i\;\; P_i\,(z)$$

$$g_n(z) = c_0\, P_0\,(z) + c_1\, P_1\,(z) + \ldots + c_n\, P_n\,(z) = \sum_{i=0}^{n} c_i\, P_i\,(z)$$

$$\int_{-1}^{1} R\,(z)\,dz = \int_{-1}^{1} \prod_{i=0}^{n} (z - z_i)\; g_n(z)\; dz = \sum_{i=0}^{n} \sum_{j=0}^{n} b_i\, c_j \int_{-1}^{1} P_i\, P_j\; dz + b_{n+1} \sum_{i=0}^{n} c_i \int_{-1}^{1} P_i\, P_{n+1}\, dz$$

Since

$$\int_{-1}^{1} P_i\,(z)\, P_j(z)\, dz = 0 \quad \text{for } i \neq j$$

$$\int_{-1}^{1} R\,(z)\,dz = \sum_{i=0}^{n} b_i\, c_i \int_{-1}^{1} [P_i\,(z)]^2\, dz$$

Since $c_i \neq 0$ and $\int_{-1}^{1} [P_i(z)]^2 \, dz \neq 0$

If $\int_{-1}^{1} R(z) \, dz = 0$ the only way is to choose $b_i = 0$ for $i = 0, 1, ..., n$.

Then $\prod_{i=0}^{n} (z - z_i) = \prod_{i=0}^{n+1} b_i \, P_i(z) = b_{n+1} \, P_{n+1}(z)$

So, choose z_i's such that $P_{n+1}(z) = 0$. i.e., z_i's are the roots of the Legendre Polynomial $P_{n+1}(z)$.

Legendre Polynomials:

$P_0 = 1$ $P_1 = z$

$P_2 = \frac{1}{2}(3z^2 - 1)$ $P_3 = \frac{1}{2}(5z^3 - 3z)$

Gauss - Legendre Quadrature Method

$$I = \int_a^b f(x)\,dx = \int_{-1}^{+1} F(z)\,dz \cong \sum_{i=1}^{N} w_i\, F(z_i) \quad , \quad N = 2, 3, ...$$

$$I \equiv \sum_{i=1}^{N} w_i\, F(z_i) \quad \text{if } F(z_i) \text{ is a polynomial of degree } 2N-1 \text{ or LESS}$$

Change of Variable:

$$x = \frac{(b-a)\,z + b + a}{2} \quad , \quad dx = \frac{b-a}{2}\,dz$$

z_i's are the roots of the Legendre polynomial, $P_N(z) = 0$

w_i's are calculated from:

$$w_i = \int_{-1}^{+1} L_i(z)\,dz = \int_{-1}^{+1} \prod_{\substack{j=1 \\ j\neq i}}^{N} \left(\frac{z - z_j}{z_i - z_j} \right) dz$$

Gauss-Legendre Nodes and Weights

N	± x	w
2	0.5773502692	1.0000000000
3	0.0000000000	0.8888888889
	0.7745966692	0.5555555556
4	0.3399810436	0.6521451549
	0.8611363116	0.3478548451
5	0.0000000000	0.5688888889
	0.5384693101	0.4786286705
	0.9061798459	0.2369268850
6	0.2386191861	0.4679139346
	0.6612093865	0.3607615730
	0.9324695142	0.1713244924
7	0.0000000000	0.4179591837
	0.4058451514	0.3818300505
	0.7415311856	0.2797053915
	0.9491079123	0.1294849662

N	± x	w
8	0.1834346425	0.3626837834
	0.5255324099	0.3137066459
	0.7966664774	0.2223810345
	0.9602898565	0.1012285363
9	0.0000000000	0.3302393550
	0.3242534234	0.3123470770
	0.6133714327	0.2606106964
	0.8360311073	0.1806481607
	0.9681602395	0.0812743884
10	0.1488743390	0.2955242247
	0.4333953941	0.2692667193
	0.6794095683	0.2190863625
	0.8650633667	0.1495513492
	0.9739065285	0.0666713443

Gauss - Legendre Quadrature Method

$$I \equiv \sum_{i=1}^{N} w_i \, F(z_i) \qquad \text{if } F\left(z_i\right) \text{ is a polynomial of degree } 2N-1 \text{ or LESS}$$

Example when N = 2:

$$P_2\left(z\right) = \frac{1}{2}\left(3 z^2 - 2\right) = 0$$

$$z_1 = -\frac{1}{\sqrt{3}} \qquad \text{and} \qquad z_2 = \frac{1}{\sqrt{3}}$$

w_i's are calculated from:

$$w_i = \int_{-1}^{+1} L_i\left(z\right) dz = \int_{-1}^{+1} \prod_{\substack{j=1 \\ j \neq i}}^{N} \left(\frac{z - z_j}{z_i - z_j}\right) dz$$

Example when N = 2:

$$w_1 = \int_{-1}^{+1} L_1(z)\,dz = \int_{-1}^{+1} \prod_{\substack{j=1\\ j\neq 1}}^{N=2} \left(\frac{z - z_j}{z_1 - z_j}\right) dz = \int_{-1}^{+1} \left(\frac{z - z_2}{z_1 - z_2}\right) dz$$

$$= \int_{-1}^{+1} \left(\frac{z - \frac{1}{\sqrt{3}}}{-\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}}}\right) dz = 1$$

$$w_2 = \int_{-1}^{+1} L_2(z)\,dz = \int_{-1}^{+1} \prod_{\substack{j=1\\ j\neq 2}}^{N=2} \left(\frac{z - z_j}{z_2 - z_j}\right) dz = \int_{-1}^{+1} \left(\frac{z - z_1}{z_2 - z_1}\right) dz$$

$$= \int_{-1}^{+1} \left(\frac{z + \frac{1}{\sqrt{3}}}{\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}}\right) dz = 1$$

Examples:

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Define a new independent variable: $z = \dfrac{2x - b - a}{b - a}$ $\quad dz = \dfrac{2}{b - a}\, dx$

Find dz and F(z) $\quad I = \int_a^b f(x)\, dx = \int_{-1}^{1} F(z)\, dz$

Choose N

Find nodes (x_i) and weights (w_i) from tabulated data

The nodes (x_i) are the zeros of **Legendre orthogonal polynomials**, which are orthogonal in the range, (-1, +1)

$$I = \int_a^b f(x)\, dx = \int_{-1}^{1} F(z)\, dz \cong \sum_{i=1}^{N} w_i\, F(z_i)$$

Error expression for Gauss-Legendre quadrature method:

$$E \cong \frac{2^{2N+1}\,[N!]^4}{(2N+1)\,[(2N)!]^3}\, f^{(2N)}(\xi) \qquad , \quad -1<\xi<1$$

where N is the number of nodes and weights used.

See the Web site «Gaussian Quadrature Weights and Abscissae» for more on Gauss-Legendre nodes and weights.

GAUSS QUADRATURE METHODS

Name	Quadrature Formula
Gauss-Legendre	$\int_{-1}^{+1} f(x)\,dx \cong \sum_{i=1}^{N} w_i\, f(x_i)$
Gauss-Chebyshev	$\int_{-1}^{+1} \frac{1}{\sqrt{1-x^2}} f(x)\,dx \cong \sum_{i=1}^{N} w_i\, f(x_i)$
Gauss-Laguerre	$\int_{0}^{\infty} e^{-x} f(x)\,dx \cong \sum_{i=1}^{N} w_i\, f(x_i)$
Gauss-Hermite	$\int_{-\infty}^{\infty} e^{-x^2} f(x)\,dx \cong \sum_{i=1}^{N} w_i\, f(x_i)$

ORTHOGONAL POLYNOMIALS

Name	Range of Orthogonality	Weighting Function	Integral Property
Legendre	[- 1 , + 1]	w(x) = 1	$\int_{-1}^{+1} w\, P_k\, P_k\, dx = \frac{2}{2k+1}$
Chebyshev	[- 1 , + 1]	$w(x) = \frac{1}{\sqrt{1-x^2}}$	$\int_{-1}^{+1} w\, T_k\, T_k\, dx = \pi \quad \text{if } k=0$ $\int_{-1}^{+1} w\, T_k\, T_k\, dx = \frac{\pi}{2} \quad \text{if } k>0$
Laguerre	[0 , ∞]	$w(x) = x^{\alpha} e^{-x}$	$\int_{0}^{\infty} w\, L_k^{\alpha}\, L_k^{\alpha}\, dx = \frac{\Gamma(\alpha+k+1)}{k!}$
Hermite	[- ∞ , +∞]	$w(x) = e^{-x^2}$	$\int_{-\infty}^{+\infty} w\, H_k\, H_k\, dx = \sqrt{\pi}\; 2^k\; k!$

Legendre Polynomials

$P_0(x) = 1$

$P_1(x) = x$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

Chebyshev Polynomials

$T_0(x) = 1$	$T_1(x) = x$	$T_2(x) = 2x^2 - 1$
$T_3(x) = 4x^3 - 3x$	$T_4(x) = 8x^4 - 8x^2 + 1$	$T_5(x) = 16x^5 - 20x^3 + 5x$

Laguerre Polynomials

$$L_0^0(x) = 1$$

$$L_1^0(x) = -x + 1$$

$$L_2^0(x) = \frac{1}{2}(x^2 - 4x + 2)$$

$$L_3^0(x) = \frac{1}{6}(-x^3 + 9x^2 - 18x + 6)$$

$$L_4^0(x) = \frac{1}{24}(x^4 - 16x^3 + 72x^2 - 96x + 24)$$

Hermite Polynomials

$H_0(x) = 1$	$H_1(x) = 2x$	$H_2(x) = 4x^2 - 2$
$H_3(x) = 8x^3 - 12x$	$H_4(x) = 16x^4 - 48x^2 + 12$	$H_5(x) = 32x^5 - 160x^3 + 120x$

GAUSS QUADRATURE METHODS

Name	Nodes and Weights		
	N	$\pm x_i$	w_i
Gauss-Legendre	2	0.577350269	1.00000000
	3	0 0.774596669	0.888888889 0.555555556
	4	0.339981043 0.861136312	0.652145155 0.347854845

Gauss-Chebyshev

$$x_i = \cos\left(\frac{i - \frac{1}{2}}{N}\pi\right) \qquad i = 1, 2, \dots N$$

$$w_i = \frac{\pi}{N} \quad \text{for all } i$$

Gauss-Laguerre

N	x_i	w_i
2	0.58578 64376 3.41421 35624	0.85355 33906 0.14644 66094
3	0.41577 45568 2.29428 03603 6.28994 50829	0.71109 30099 0.27851 77336 0.01038 92565
4	0.32254 76896 1.74576 11012 4.53662 02969 9.39507 09123	0.60315 41043 0.35741 86924 0.03888 79085 0.00053 92947

Gauss-Hermite

N	$\pm x_i$	w_i
2	0.70710 67811	0.88622 69255
3	0. 1.22474 48714	1.18163 59006 0.29540 89752
4	0.52464 76233 1.65068 01239	0.80491 40900 0.08131 28354
5	0. 0.95857 24646 2.02018 28705	0.94530 87205 0.39361 93232 0.01995 32421