Linear, first-order PDEs

$$\frac{du(x)}{dx} = u_x = 0 \qquad \text{If ODE, the solution is:} \quad u = C$$

$$\frac{\partial u(x,y)}{\partial x} = u_x = 0 \qquad \text{If PDE, the solution is:} \quad u = f(y)$$

Notice that where the solution of an ODE contains arbitrary constants, the solution to a PDE contains arbitrary functions.

Any linear, first-order PDE can be reduced to an ordinary differential equation, which will then allow as to tackle it with already familiar methods from ODEs.

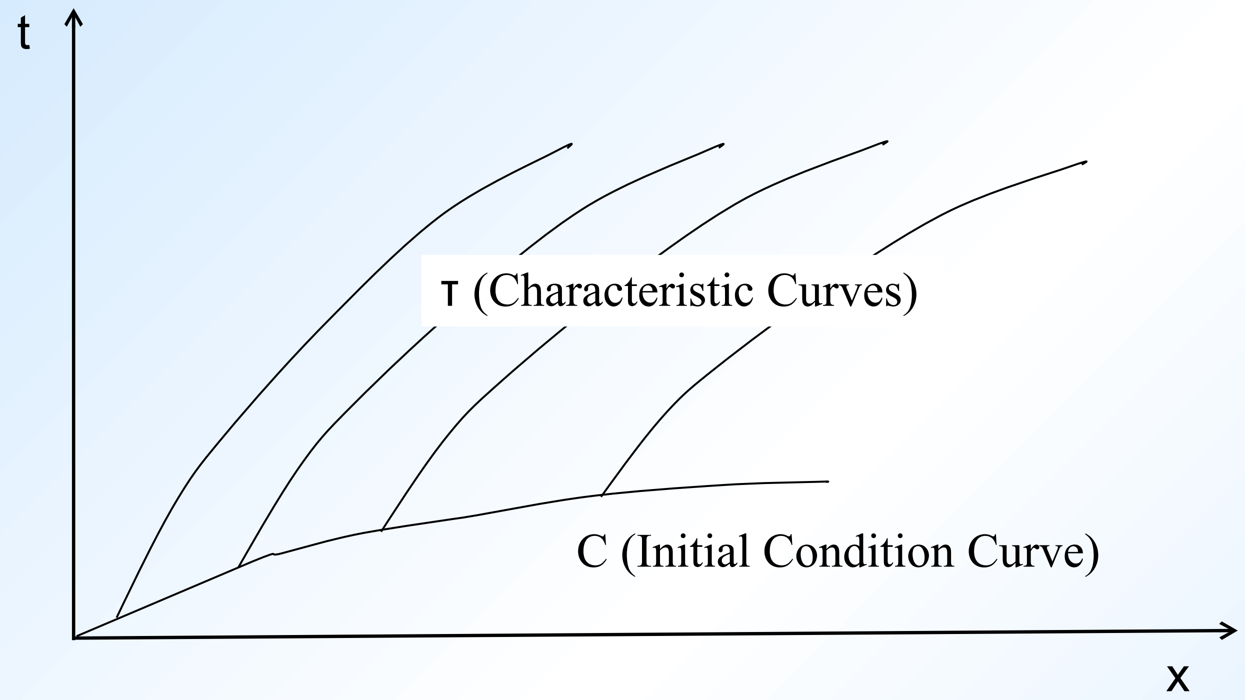
This method of reducing the PDE to an ODE is called **the method of characteristics**.

The Method of Characteristics

Consider the quasi-linear, first order PDE: $$a(x,t,u)\frac{\partial u}{\partial x} + b(x,t,u)\frac{\partial u}{\partial t} = c(x,t,u)$$

For a solution, u(x,t), satisfying the PDE to be possible, we require that the values of u be specified along some initial curve, C. In many cases, C will be a part of x-axis, and then u will be given as a function of x on t = 0

It is anticipated that the PDE will have to be solved by numerical integration and it would be convenient to integrate in one direction only with no disturbances from the derivatives in other directions. Thus, we seek a curve, τ, called the **characteristic curve** in the x-t plane along which this can be achieved.



Suppose such a curve, τ, exists, and is given parametrically as $x = x(s)$ and $t = t(s)$

So, the the PDE reduces to a function of s, only: $u(s) = u(x(s)\,,\,t(s))$

Given PDE: $$a \frac{\partial u}{\partial x} + b \frac{\partial u}{\partial t} = c \quad \ldots\ldots\ldots\ldots \quad (1)$$

If the parameter s exists: $$\frac{du}{ds} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial t}\frac{dt}{ds} \quad \ldots\ldots\ldots\ldots \quad (2)$$

Eliminate $\frac{\partial u}{\partial x}$ from (1) and (2):

$$c \frac{dx}{ds} - a \frac{du}{ds} = \left(b \frac{dx}{ds} - a \frac{dt}{ds} \right) \frac{\partial u}{\partial t} \quad \ldots\ldots\ldots\ldots \quad (3)$$

The effect of $\frac{\partial u}{\partial t}$ can be eliminated if $$b \frac{dx}{ds} = a \frac{dt}{ds} = \lambda \quad \ldots\ldots\ldots\ldots \quad (4)$$

Equations (3) and (4) give: $$\frac{dx}{a} = \frac{dt}{b} = \frac{du}{c}$$

$$\frac{dx}{a} = \frac{dt}{b} = \frac{du}{c}$$

$$\frac{dx}{a} = \frac{dt}{b}$$

This is the **characteristic equation**, which gives a relation between independent variables, x and t.

$$c\,dx = a\,du$$

The remaining part is called the **solution equation**, which is the given PDE, true when the characteristic equation exists.

Note that the parameter, s (a relation between independent variables, x and t) exists when the PDE is linear and first-order.

Example: $\dfrac{\partial u}{\partial x} + 2\dfrac{\partial u}{\partial t} = 1$ a = 1 , b = 2 , c = 1

$$\frac{dx}{a} = \frac{dt}{b} \quad \Rightarrow \quad \frac{dx}{1} = \frac{dt}{2} = \frac{du}{1}$$

Characteristic Equation: $\dfrac{dt}{dx} = 2 \quad \Rightarrow \quad t = 2x + A$ where A is a constant

Characteristics curves are a family of straight lines

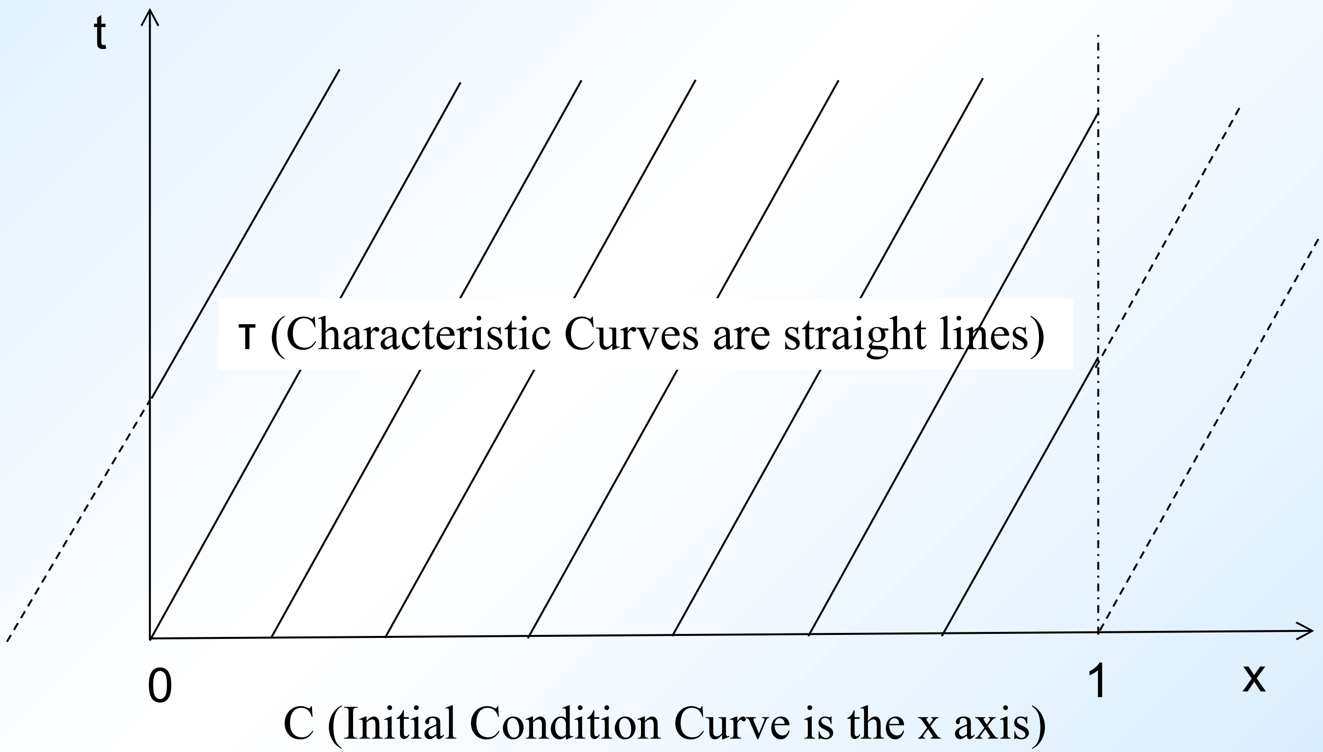
Solution Equation: $dx = du \quad \Rightarrow \quad u = x + B$ where B is another constant

Along any characteristic line, $t = 2x + A$, the value of u is given by $u = x + B$

If the given Initial Condition is $u(x,0) = x(x-1)$ for $0 \le x \le 1$ and $t = 0$

Characteristic Equation: $t = 2x - A$

Solution Equation: $u = x + B$ Apply IC => $u = x + \frac{A}{4}(A-4)$



Note that if $A < 0$ or $A > 2$, the corresponding characteristic line cuts the x axis beyond $0 \leq x \leq 1$ where the initial condition is not specified. That is we cannot obtain the solution outside the bounding characteristics at $A = 0$ and $A = 2$, or $t = 2x$ and $t = 2x - 2$

Example: Suppose the Initial Condition is

$$u(x,0) = x \quad \text{for} \quad 0 \leq x \leq 1/2 \quad \text{and} \quad t = 0$$
$$= 0 \quad \text{for} \quad 1/2 \leq x \leq 1 \quad \text{and} \quad t = 0$$

In this case, there is a discontinuity at $u(1/2,0)$ which will persists along the characteristics, $t = 2x – 1$. We cannot obtain a solution along this characteristics.

Transport Equation

A particular example of a first-order, constant-coefficient, linear equation is the transport, or advection equation (first-order wave equation)

$$a \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0$$

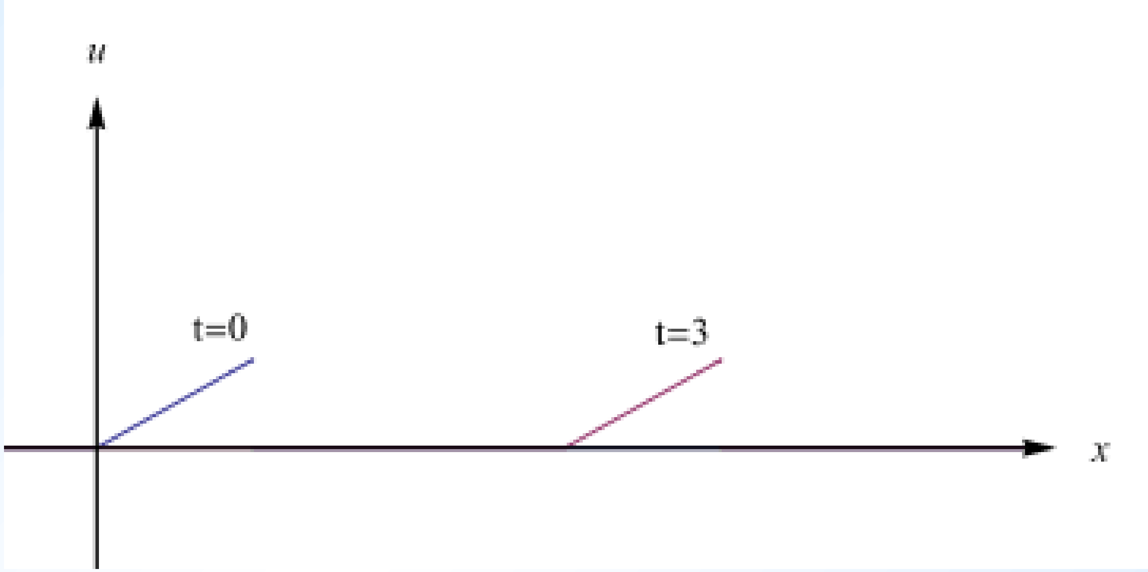
which describes motions with constant speed, a.

This equation can be used to model air pollution, dye dispersion, or even traffic flow with u representing the density of the pollutant (or dye or traffic) at position x and time t.

Characteristic Equation: $\frac{dx}{dt} = a \quad \Rightarrow \quad x = a\,t + x(0)$

$$t = \frac{1}{a}\left[x - x(0)\right]$$

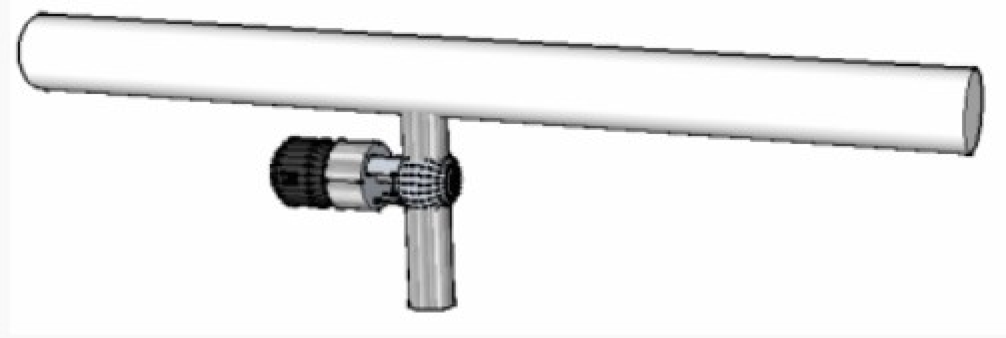
Solution Equation: $\frac{du}{dx} = 0 \quad \Rightarrow \quad u = \text{constant}$



This says that the value of u remains constant along each characteristic straight line:

$$u(x,t) = u(x(0),0)$$

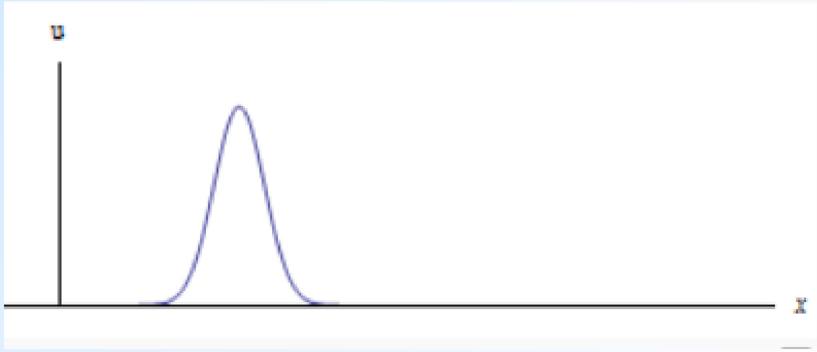
Example:



Imagine a device that consists of a pipe with constant radius r and a valve in which a chemical can be injected. This pipe contains a fluid, for example water.

Assume that once we injected the chemical, the concentration varies only in the direction of the pipe. Further, we assume that the diffusion is negligible and that the concentration profile in the pipe that results form the injection at $x = x_0$ has a bell shape

$$u(x,0) = e^{-x^2} \quad \text{at } t = 0$$



$u(x,0) = e^{-x^2}$ at $t = 0$

Now we ask, how the fluid containing the chemical moves forward, when pressure is applied so that the fluid starts moving. Since there is no diffusion and no turbulences in the pipe the package of fluid containing the chemical is shifted along the pipe with some velocity v. The bell shape thereby is preserved.

After time Δt, the package was transported the distance $\Delta x = v\,\Delta t$ in the direction v. So, we can give the concentration profile u(x) after time Δt as simply shifting the bell-shaped curve by Δx to the right.

This can be done by adding (- v Δt) to the independent variable x so we get

$$e^{-(x - x_0)^2} = e^{-(x - v\,\Delta t)^2}$$

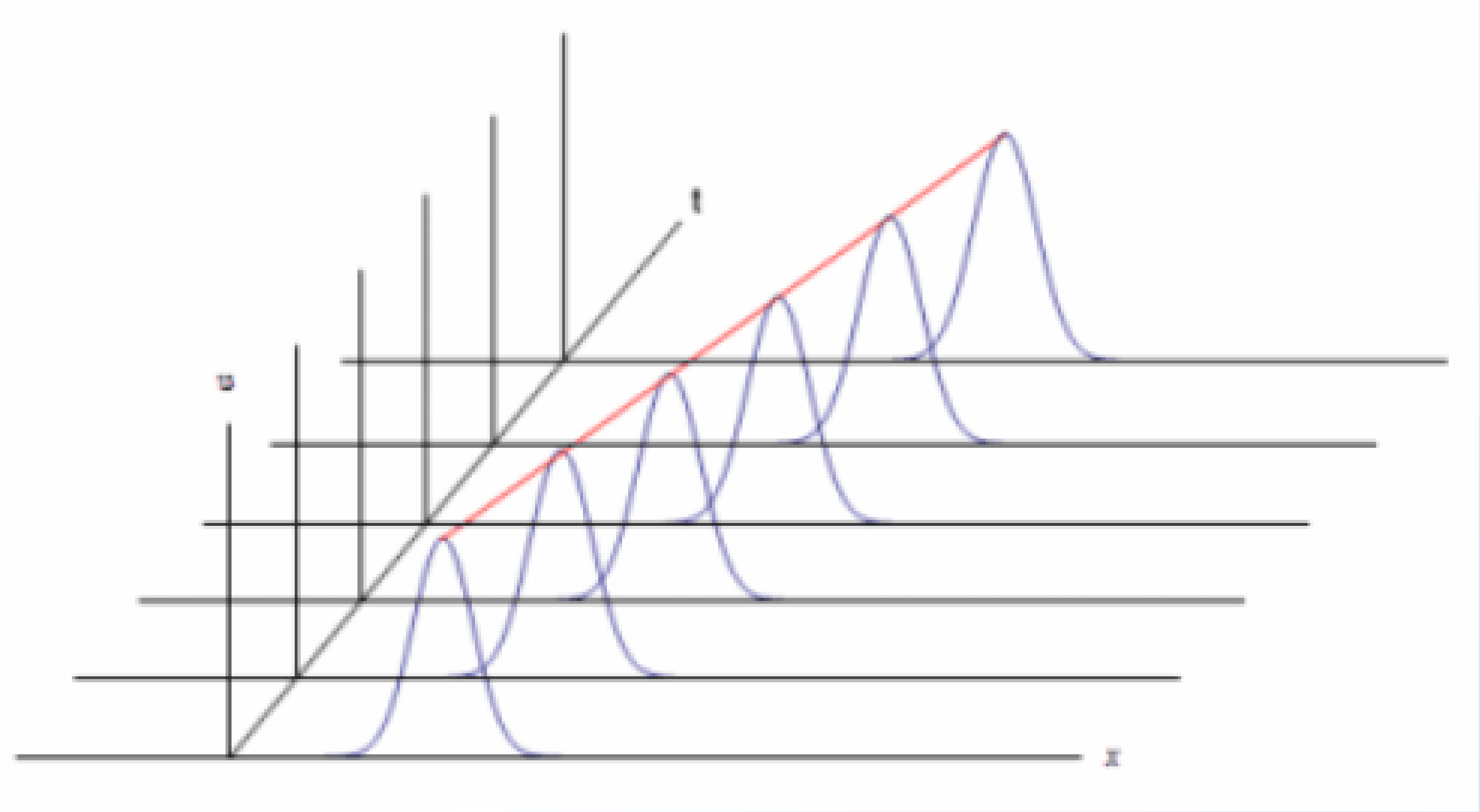
For some arbitrary time t, the displacement of the concentration profile can be written as

$$u(x,t) = e^{-(x - v\,t)^2}$$

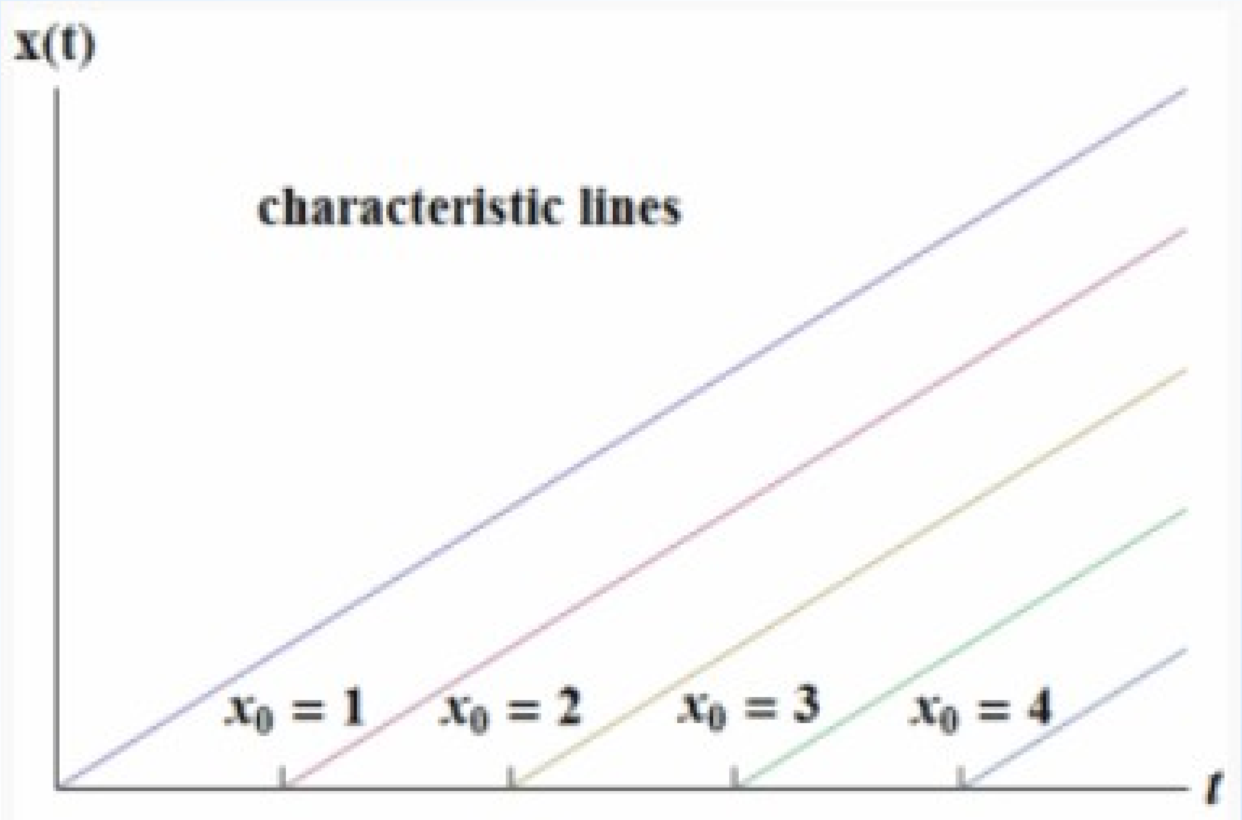
This equation describes a concentration profile u(x) travelling with speed v in the positive x direction.

The plot the the concentration as a function of space and time:

$$u(x,t) = e^{-(x - v\,t)^2}$$



$$u(x,t) = e^{-(x - v\,t)^2}$$



Non-Linear, first-order PDEs

In the previous example, we were able to solve (integrate)

$$\frac{dt}{dx} = \frac{b}{a} = 2 \qquad \text{and} \qquad \frac{du}{dx} = \frac{c}{a} = 1$$

analytically (exactly). This is not always so, and we have to resort to numerical integration of the characteristic equation and the solution equation.

Example: $$x\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial t} = x + t \qquad a = x \;,\; b = u \;,\; c = x + t$$

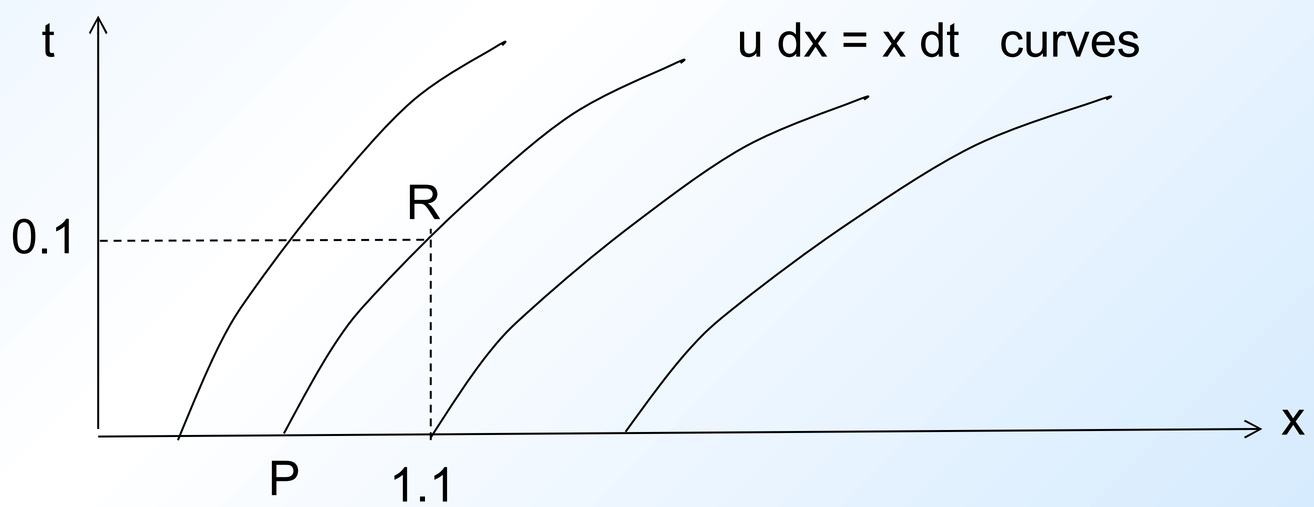
Initial Condition: u(x,0) = 1 at t = 0, any x

Question: Find u(1.1 , 0.1)

$$\frac{dx}{a} = \frac{dt}{b} = \frac{du}{c} \quad \Rightarrow \quad \frac{dx}{x} = \frac{dt}{u} = \frac{du}{x+t}$$

Characteristic equation: $\frac{dx}{x} = \frac{dt}{u} \quad \Rightarrow \quad u\,dx = x\,dt$

Solution equation: $\frac{dx}{x} = \frac{du}{x+t} \quad \Rightarrow \quad x\,du = (x+t)\,dx$



Integrate both the characteristic equation and the solution equation along the characteristic curve:

Charac. Eq'n: $$\int_{t_P}^{t_R} x\,dt = \int_{x_P}^{x_R} u\,dx \qquad \int_{0}^{0.1} x\,dt = \int_{x_P}^{1.1} u\,dx$$

Solution Eq'n: $$\int_{x_P}^{x_R} (x+t)\,dx = \int_{u_P}^{u_R} x\,du \qquad \int_{x_P}^{1.1} (x+t)\,dx = \int_{1}^{u_R} x\,du$$

The point R is known, but the point P where the characteristic curve intersects the x-axis is not known. $u_P = 1$ is the given initial condition.

When the point R is close to the point P, one may assume linear variation along the characteristic curve.

Charac. Eq'n: $$\left(\frac{x_R + x_P}{2}\right)(t_R - t_P) = \left(\frac{u_R + u_P}{2}\right)(x_R - x_P)$$

Solution Eq'n: $$\left(\frac{x_R + t_R + x_P + t_P}{2}\right)(x_R - x_P) = \left(\frac{x_R + x_P}{2}\right)(u_R - u_P)$$

Substitute known values:

Charac. Eq'n: $(1.1 + u_R)\, x_P = 1.1\, u_R + 0.99$

Solution Eq'n: $(1.1 + x_P)\, u_R = (1.2 + x_P)\,(1.1 - x_P) + (1.1 + x_P)$

Two non-linear equations in two unknowns. Solve by iteration. Start with $u_P = 1$

x_P		u_R		x_P
		1.0000	=>	0.9952
0.9952	=>	1.1098	=>	1.0004
1.0004	=>	1.1043	=>	1.0002
1.0002	=>	1.1046	=>	1.0002
1.0002	=>	1.1045	=>	
….		….		….
		1.105		

$u(1.1, 0.1) = 1.105$ correct to three decimal places.

This method of characteristics can be extended to system of first-order PDE's as well as second-order PDE's of certain kind (hyperbolic).