NON-LINEAR SYSTEM OF EQUATIONS

Solution with **fixed-point iterations**:

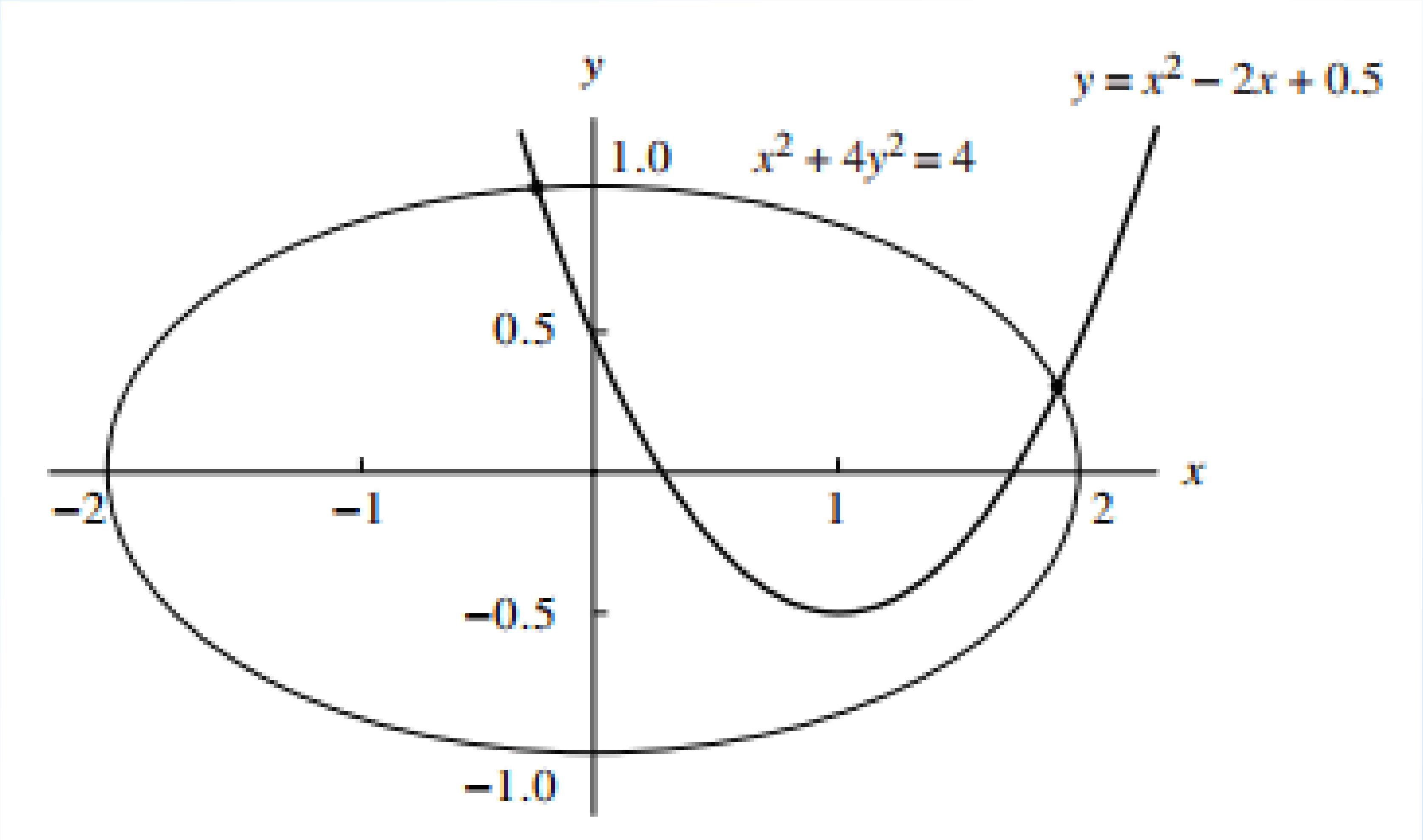
Given : A parabola	$f_1(x, y) = x^2 - 2x - y + 0.5 = 0$
and an ellipse	$f_2(x, y) = x^2 + 4y^2 - 4 = 0$

Convert to

$$x = \frac{x^2 - y + 0.5}{2} = g_1(x, y)$$

$$y = \frac{-x^2 - 4y^2 + 8y + 4}{8} = g_2(x, y)$$

Add 8y to both sides of the equation f_2 and solve for y.



y
y = x^2 − 2x + 0.5
1.0
x^2 + 4y^2 = 4
0.5
x
−2
−1
1
2
−0.5
−1.0

Start with (0,1)		
n	x_n	y_n
0	0	1
1	- 0.25	1
2	- 0.21875	0.99219
3	- 0.22217	0.99399
4	- 0.22231	0.99381
5	- 0.22219	0.99380
6	- 0.22222	0.99381
7	- 0.22221	0.99381

Start with (2,0)		
n	x_n	y_n
0	2	0
1	2.25	0
2	2.78125	- 0.13228
3	4.18408	- 0.60855
4	9.30755	- 2.48204
5	44.80623	- 15.89109
6	1011.995	- 392.6042
7	512263.2	- 205477.8

Given : A parabola	$f_1(x, y) = x^2 - 2x - y + 0.5 = 0$
and an ellipse	$f_2(x, y) = x^2 + 4y^2 - 4 = 0$

Add -2x to f_1 and -11y to f_2 and solve for x and y:

$$x = \frac{-x^2 + 4x + y - 0.5}{2} = g_1(x, y)$$

$$y = \frac{-x^2 - 4y^2 + 11y + 4}{11} = g_2(x, y)$$

Theorem on convergence in fixed-point iteration:

Let the functions, $g_1(x,y)$ and $g_2(x,y)$ and their first partial derivatives be continuous on a region that contains the fixed point. Assume that the starting point is chosen sufficiently close to the fixed point and

$$\left| \frac{\partial}{\partial x} g_1(x,y) \right| + \left| \frac{\partial}{\partial x} g_2(x,y) \right| < 1 \qquad \text{and}$$

$$\left| \frac{\partial}{\partial y} g_1(x,y) \right| + \left| \frac{\partial}{\partial y} g_2(x,y) \right| < 1$$

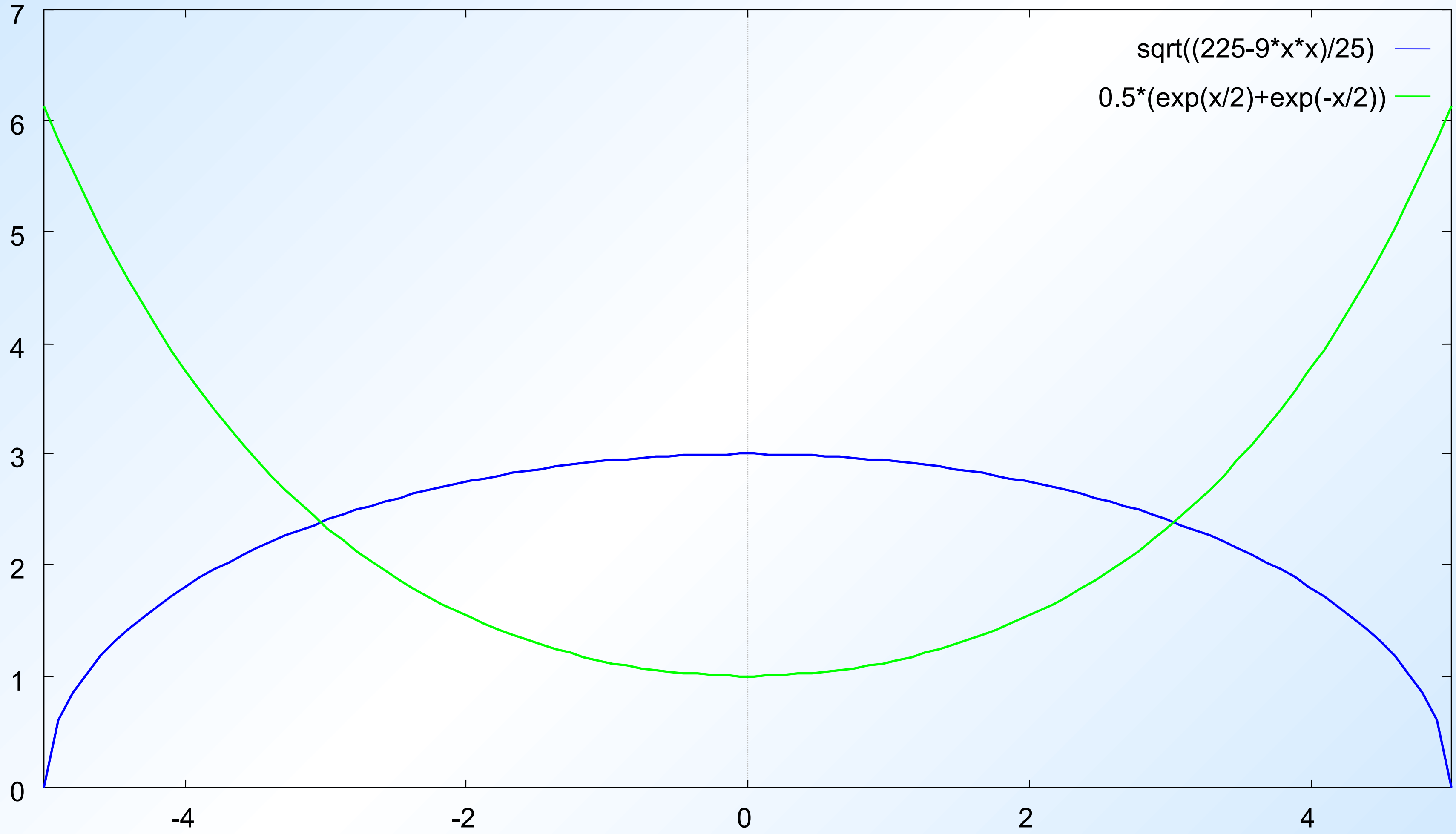
then, the iterations converge to the fixed point.

n	x_n	y_n
0	2.0000000	0.0000000
1	1.7500000	0.0000000
2	1.7187500	0.0852273
3	1.7530629	0.1776676
4	1.8083448	0.2504410
5	1.8568547	0.2939870
6	1.8867482	0.3127486
7	1.8999613	0.3171972

Solution with **Newton-Raphson (Newton's method)**:

Given : A catenary	$f_1(x,y) = y - 0.5\,(e^{x/2} + e^{-x/2}) = 0$
and an ellipse	$f_2(x,y) = 9\,x^2 + 25\,y^2 - 225 = 0$

Use Newton's method to determine the point of intersection of the curves that resides in the first quadrant of the coordinate system. Starting with the initial guess, $(x_1,y_1) = (2.5,2.0)$,


7
6
5
4
3
2
1
0
-4
-2
0
2
4
sqrt((225-9*x*x)/25)
0.5*(exp(x/2)+exp(-x/2))

Expand both non-linear functions of two variables in Taylor series around a chosen point (x_1,y_1) (as close to the real solution as possible).

Taylor Series expansion of a function of two variables, f(x,y), around (x_0,y_0)

$$f(x,y) = f(x_1,y_1) + (x - x_1)\left.\frac{\partial f}{\partial x}\right|_{x_1,y_1} + (y - y_1)\left.\frac{\partial f}{\partial y}\right|_{x_1,y_1} +$$

$$\frac{1}{2\,!}\left[(x - x_1)^2\left.\frac{\partial^2 f}{\partial x^2}\right|_{x_1,y_1} + (y - y_1)^2\left.\frac{\partial^2 f}{\partial y^2}\right|_{x_1,y_1} + 2\,(x - x_1)\,(y - y_1)\left.\frac{\partial^2 f}{\partial x\,\partial y}\right|_{x_1,y_1}\right]$$

$$+ \ldots$$

Note that these expansions are valid within a radius of convergence, r.

$$\left.\frac{\partial f_1}{\partial x}\right|_{x_1,y_1} \Delta x + \left.\frac{\partial f_1}{\partial y}\right|_{x_1,y_1} \Delta y = -\, f_1(x_1,y_1)$$

$$\left.\frac{\partial f_2}{\partial x}\right|_{x_1,y_1} \Delta x + \left.\frac{\partial f_2}{\partial y}\right|_{x_1,y_1} \Delta y = -\, f_2(x_1,y_1)$$

or

$$\underbrace{\begin{pmatrix} \left.\frac{\partial f_1}{\partial x}\right|_{x_1,y_1} & \left.\frac{\partial f_1}{\partial y}\right|_{x_1,y_1} \\ \left.\frac{\partial f_2}{\partial x}\right|_{x_1,y_1} & \left.\frac{\partial f_2}{\partial y}\right|_{x_1,y_1} \end{pmatrix}}_{\text{Jacobian}} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} -\, f_1(x_1,y_1) \\ -\, f_2(x_1,y_1) \end{pmatrix} \quad \text{or} \quad J(f_1,f_2) \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} -\, f_1(x_1,y_1) \\ -\, f_2(x_1,y_1) \end{pmatrix}$$

Karl Gustav Jacobi (1804—1851) was born in Potsdam, Prussia (now part of Germany) and was educated at the University of Berlin. He earned his doctorate in 1825 and had a position at the University of Königsberg from 1826 until 1844, after which he returned to Berlin where he remained until his death. Jacobi founded what we know as elliptic function theory and studied the determinant of functions known as the Jacobian. His iterative method for solving linear systems was first published in an astronomical journal in 1845.

$$\frac{\partial f_1}{\partial x} = -\frac{1}{4}\left(e^{x/2} - e^{-x/2}\right) \qquad \frac{\partial f_1}{\partial y} = 1 \qquad \frac{\partial f_2}{\partial x} = 18\ x \qquad \frac{\partial f_2}{\partial y} = 50\ y$$

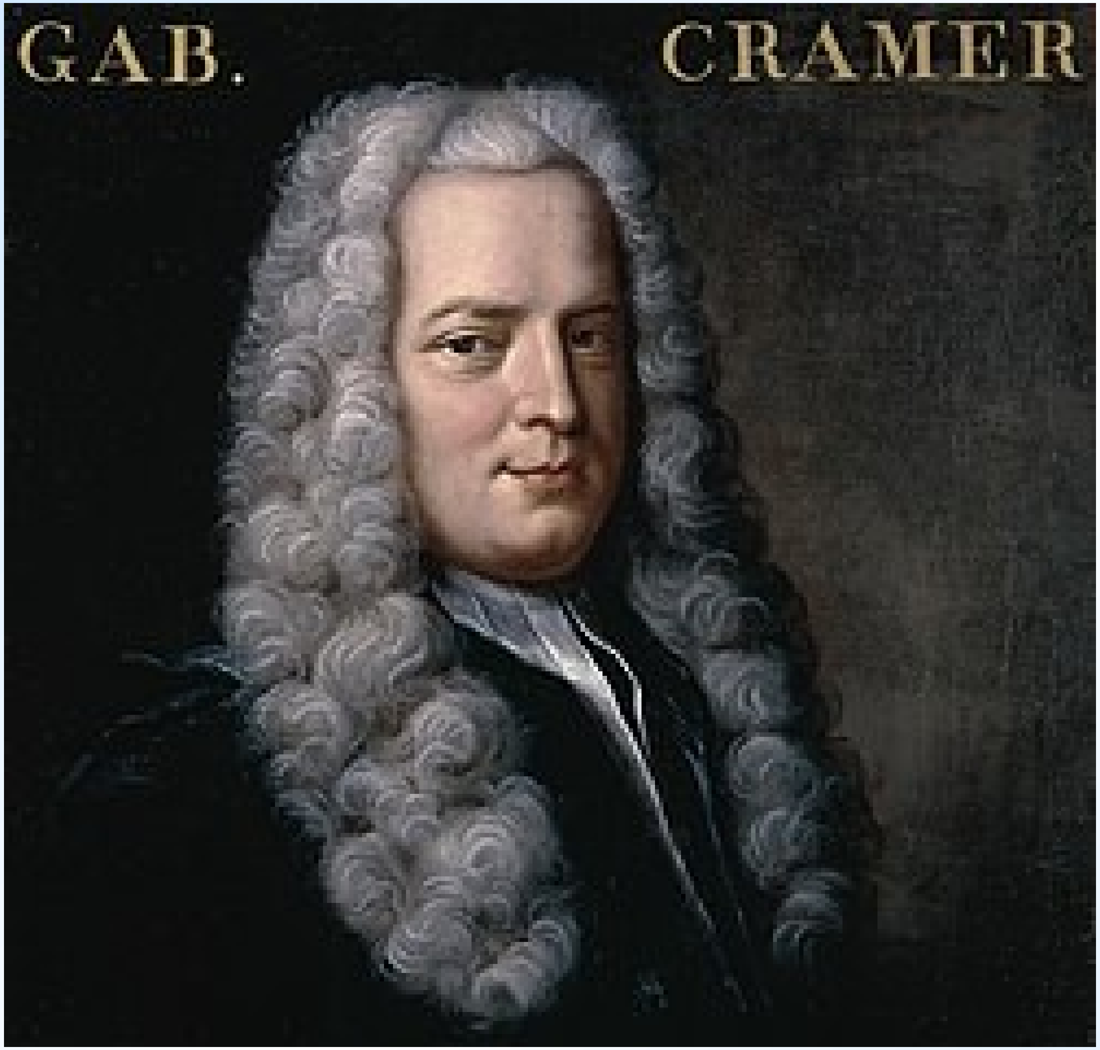
One may use Cramer's rule to solve the two equations:

$$\Delta x = \frac{-f_1(x_1,y_1)\left.\dfrac{\partial f_2}{\partial y}\right|_{x_1,y_1} + f_2(x_1,y_1)\left.\dfrac{\partial f_1}{\partial y}\right|_{x_1,y_1}}{\det\left[J(f_1,f_2)\right]}$$

$$\Delta y = \frac{-f_2(x_1,y_1)\left.\dfrac{\partial f_1}{\partial x}\right|_{x_1,y_1} + f_1(x_1,y_1)\left.\dfrac{\partial f_2}{\partial x}\right|_{x_1,y_1}}{\det\left[J(f_1,f_2)\right]}$$

where

$$\det\left[J(f_1,f_2)\right] = -\frac{1}{4}\left(e^{x_1/2} - e^{x_1/2}\right) 50\ y_1 - 18\ x_1$$



Gabriel Cramer

Genevan Mathematician

1704 -1752

x_n	y_n	$f_1(x,y)$	$f_2(x,y)$	df_1/dx	df_1/dy	df_2/dx	df_2/dy	det(J)	Δx	Δy
2.5000	2.0000	0.11157	-68.7500	-0.8009	1.0000	45.0000	100.0000	-125.095	0.6387	0.4000
3.1387	2.4000	-0.10588	7.67331	-1.1488	1.0000	56.4978	120.0026	-194.366	-0.1048	-0.0145
3.0339	2.3854	-0.00338	0.10425	-1.0847	1.0000	54.6105	119.2737	-183.991	-0.0027	0.0003
3.0311	2.3858	-0.00000	0.00007	-1.0830	1.0000	54.5608	119.2932	-183.766	-0.0000	0.0000
3.0311	2.3858	-1.0E-12	3.3E-11	-1.0830	1.0000	54.5607	119.2932	-183.766	-8.E-13	1.E-13

Example:

Given : A parabola	$f_1(x\,,\,y) = x^2 - 2\,x - y + 0.5 = 0$
and an ellipse	$f_2\,(x\,,\,y) = x^2 + 4\,y^2 - 4 = 0$

Newton-Raphson solution:

$$\begin{pmatrix} \left.\dfrac{\partial f_1}{\partial x}\right|_{x_1,y_1} & \left.\dfrac{\partial f_1}{\partial y}\right|_{x_1,y_1} \\ \left.\dfrac{\partial f_2}{\partial x}\right|_{x_1,y_1} & \left.\dfrac{\partial f_2}{\partial y}\right|_{x_1,y_1} \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} -\,f_1(x_1,y_1) \\ -\,f_2(x_1,y_1) \end{pmatrix} \qquad \text{or} \qquad J(x_1,y_1) \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} -\,f_1(x_1,y_1) \\ -\,f_2(x_1,y_1) \end{pmatrix}$$

Start with $x_1 = 2.0$ and $y_1 = 0.25$

$$J(x_1,y_1) = \begin{pmatrix} 2x_1 - 2 & -1 \\ 2x_1 & 8y_1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = - \begin{pmatrix} 0.25 \\ 0.25 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} -0.09375 \\ 0.0625 \end{pmatrix}$$

$x_2 = x_1 + (-0.09375) = 1.90625$ $\qquad$ $y_2 = y_1 + 0.0625 = 0.3125$

$x_3 = 1.900691$ $\qquad$ $y_3 = 0.311213$

$x_4 = 1.900677$ $\qquad$ $y_4 = 0.311219$