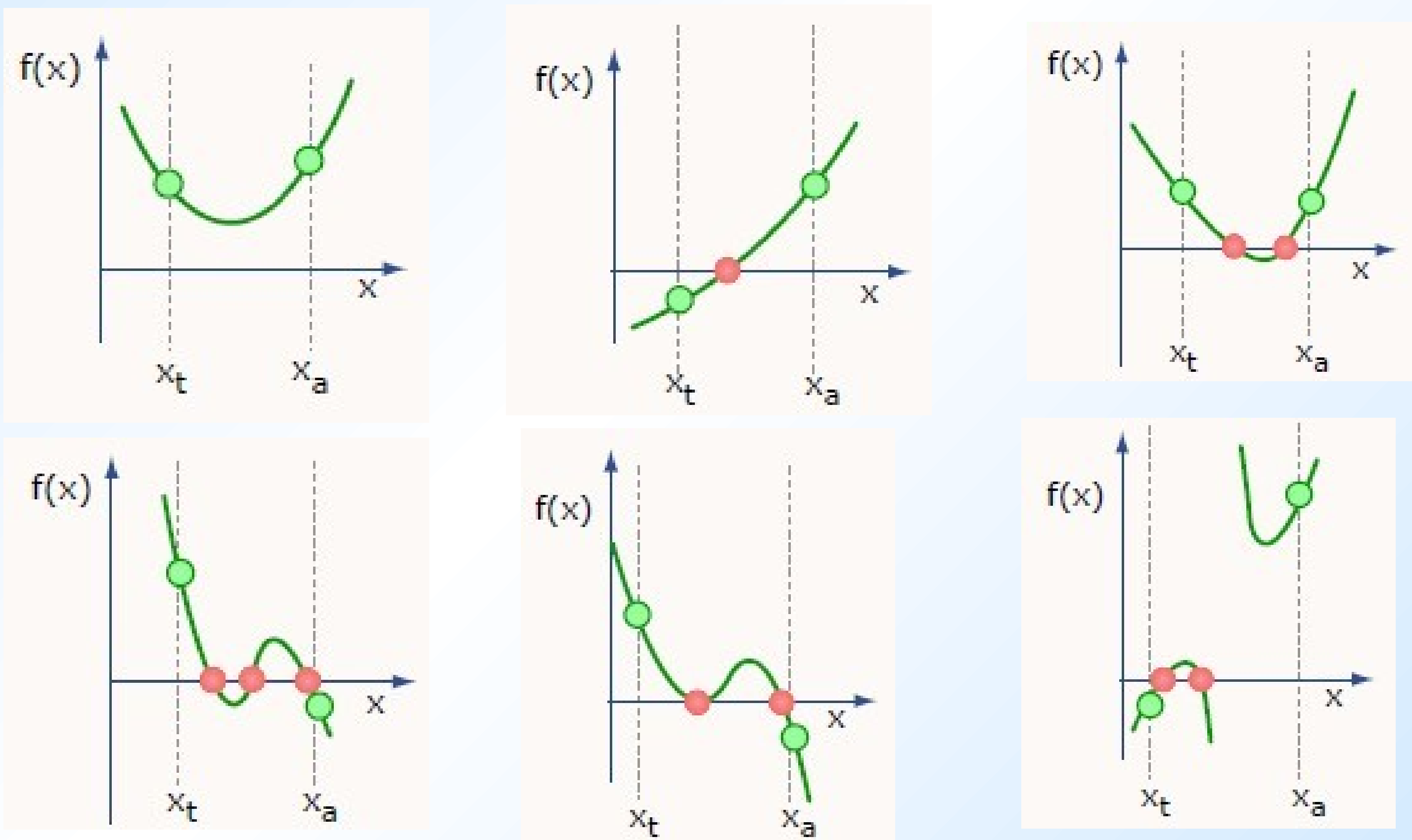
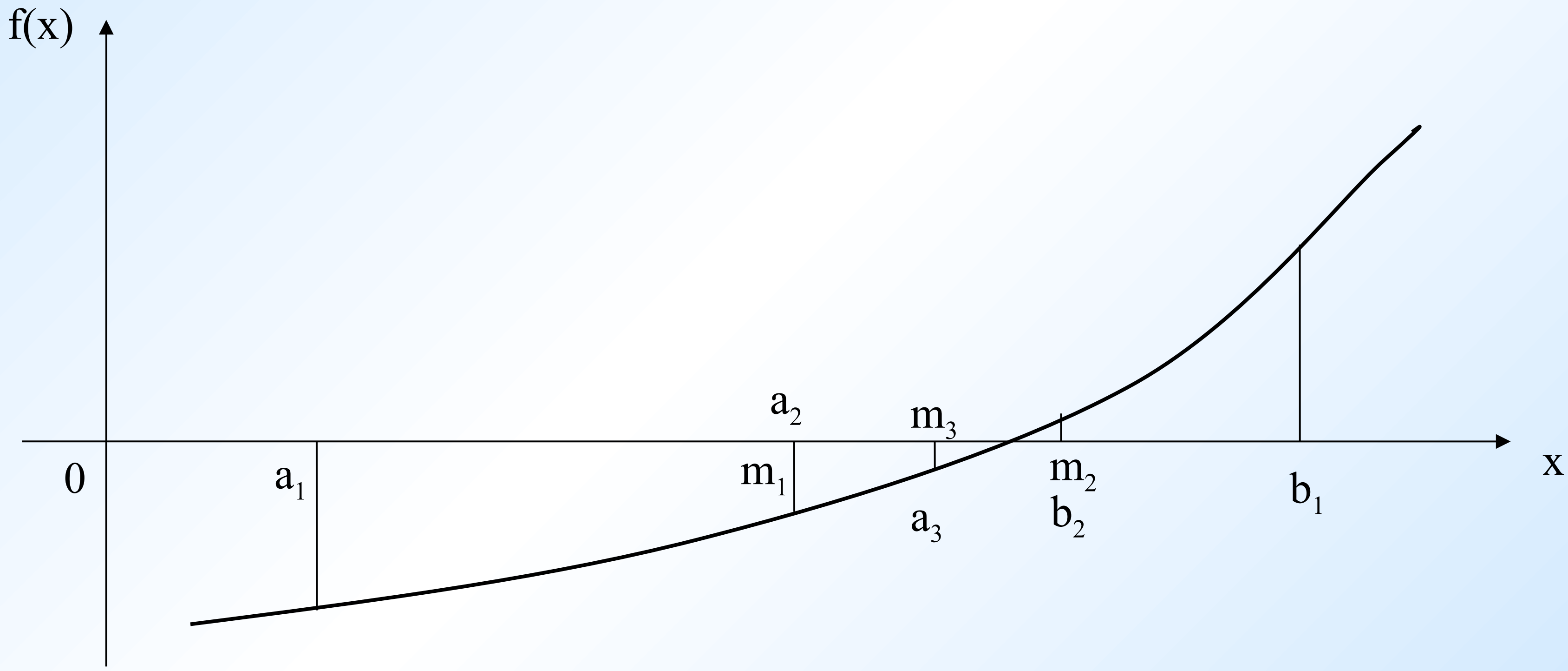
ROOTS of an Equation
(Zeros of a Function)

Given $f(x) = 0$ Question : $x = ?$ Or x's $= ?$ Several Possibilities:



BISECTION

(INTERVAL HALVING; BINARY CHOPPING; BOLZANO'S METHOD)

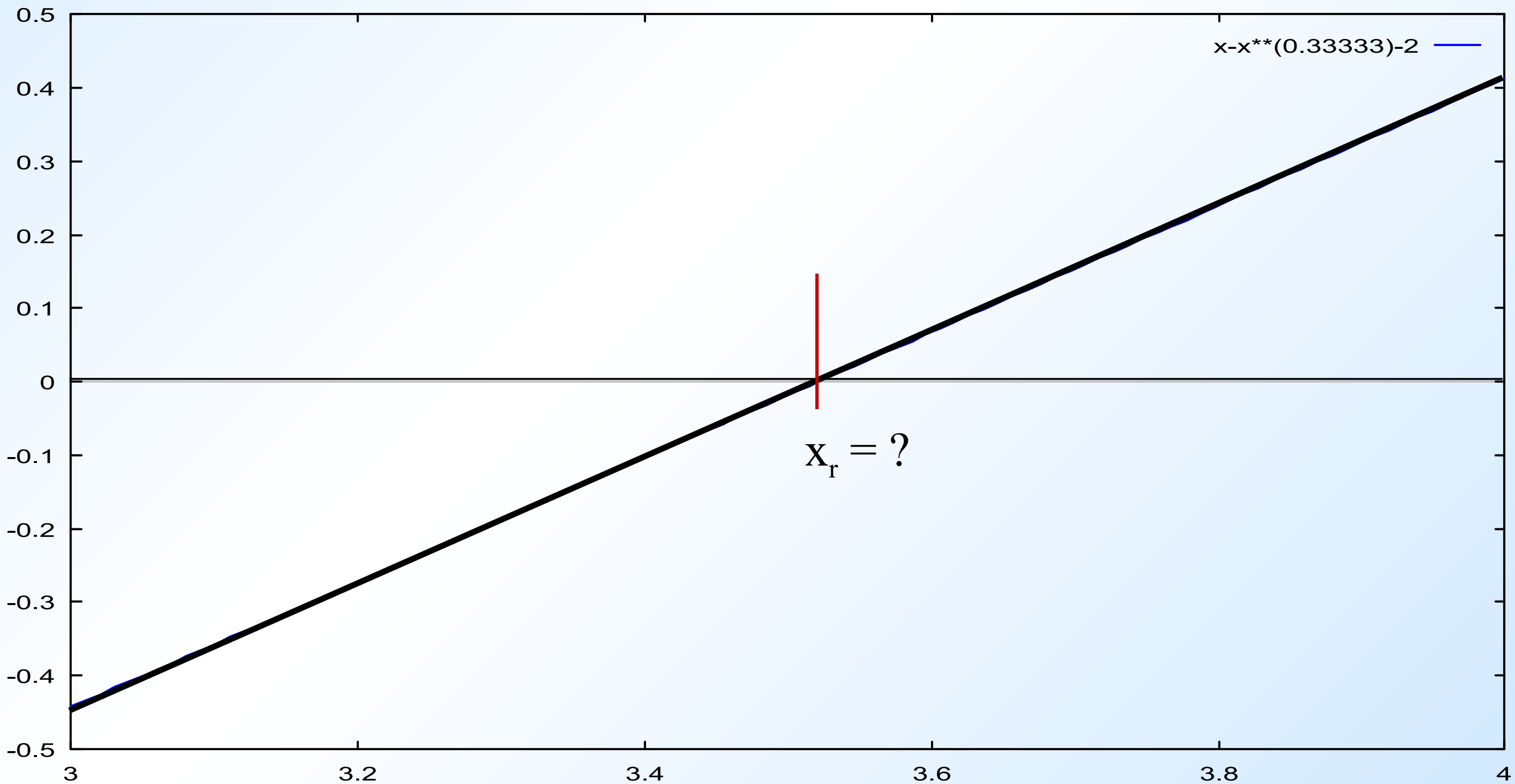


Condition:	f(x) is continuous in (a_1 , b_1) and		
	$f(a_1)\ f(b_1) < 0$, only once		
Procedure:	Let n = 1, 2, 3, …		
	Set $m_n = (a_n + b_n) / 2$		
	If $f(a_n)\ f(m_n) < 0$ …………	Set	$a_{n+1} = a_n$
			$b_{n+1} = m_n$
	If not ………………………	Set	$a_{n+1} = m_n$
			$b_{n+1} = b_n$
	f(x) has a root in (a_{n+1} , b_{n+1})		Reset m_n and repeat above

EXAMPLE on BISECTION METHOD

Find the root of $f(x) = x - x^{1/3} - 2 = 0$

in the interval [3,4], with bisection.



n	a	b	m	f (m)
0	3	4		
1	3	4	3.5	- 0.01829449
2	3.5	4	3.75	0.19638375
3	3.5	3.75	3.625	0.08884159
4	3.5	3.625	3.5625	0.03522131
5	3.5	3.5625	3.3125	0.00845016
6	3.5	3.53125	3.515625	- 0.00492550
7	3.51625	3.53125	3.5234375	0.00176150
8	3.51625	3.5234375	3.51953125	- 0.00158221
9	3.51953125	3.5234375	3.52148438	0.00008959
10	3.51953125	3.52148438	3.52050781	- 0.00074632

Stopping Criteria for Iterations

$$\left|E_{true}\right| = \left|\frac{x_r - m_{est}}{x_r}\right|$$

This is true relative (percent) error which is unknown

$$\left|E_{est}\right| = \left|\frac{m_{new\ est} - m_{old\ est}}{m_{new\ est}}\right|$$

This is estimated relative (percent) error which can be found

$$\left|E_{est}\right| = \left|m_{new\ est} - m_{old\ est}\right|$$

This is estimated error which can be found

Criterion 1 $\left|E_{est}\right| \leq \left|E_{predetermined\ limit}\right| = \varepsilon_1$

Criterion 2 $f\,(m_{new\ est}) \leq \varepsilon_2$

Criterion 3 Stop after N number of iterations where $N < \infty$

Use them all

BISECTION

(INTERVAL HALVING; BINARY CHOPPING; BOLZANO'S METHOD)

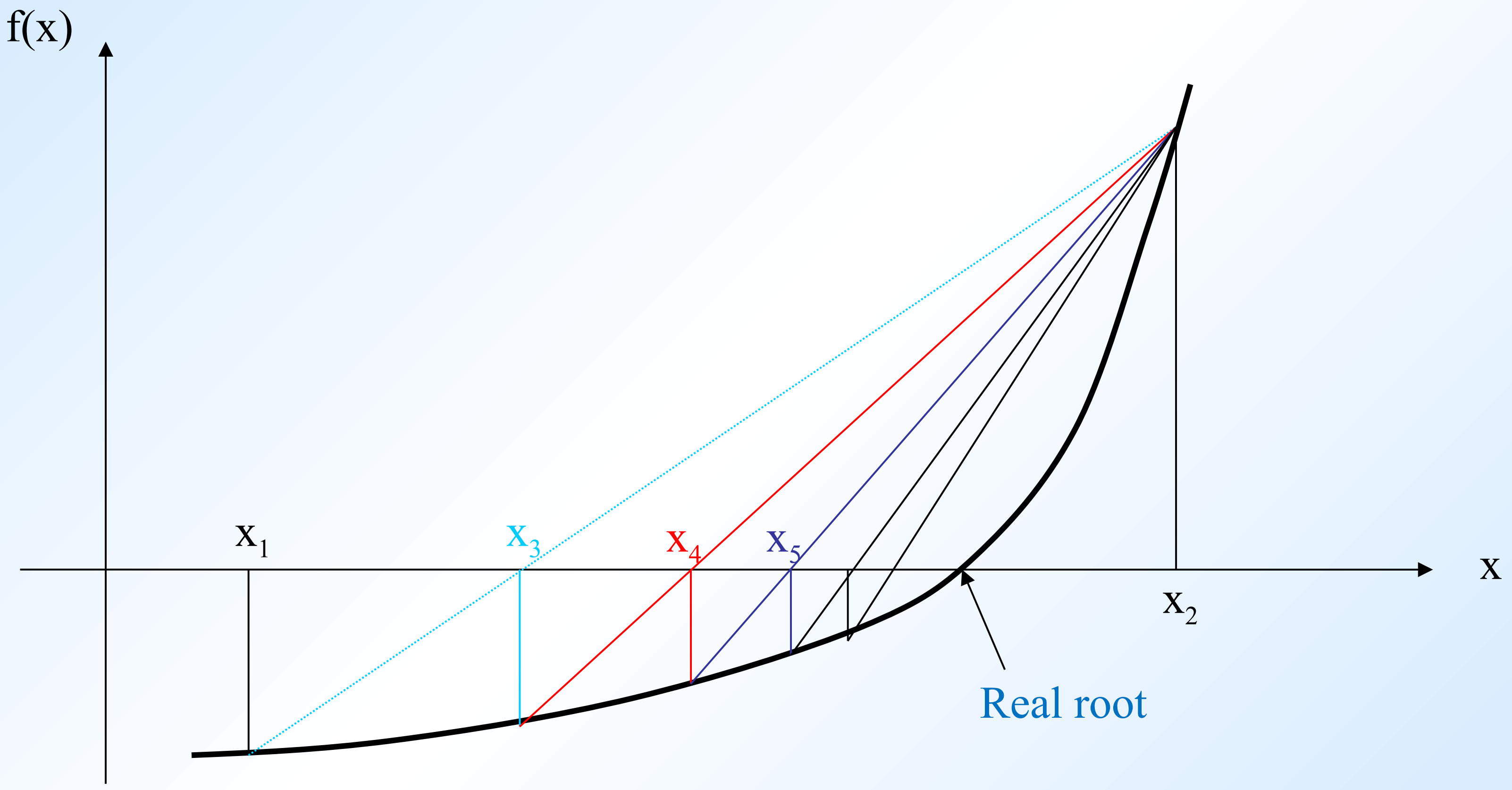
Advantages: 1. It is simple

2. The upper bound of the error is known: $|E_n| < \left| \dfrac{b_n - a_n}{2} \right|$

Disadvantage: Large number of iterations may be necessary, i.e., n can be large

i.e., time efficiency may be low

FALSE POSITION
(REGULA FALSI; LINEAR INTERPOLATION)



Condition: f (x) is continuous in (x_1, x_2) and $f(x_1)\, f(x_2) \le 0$

Procedure (Algorithm):

Assume	f(x) is linear over (x_1, x_2)
Then	$x_3 = x_2 - \dfrac{(x_2 - x_1)\, f(x_2)}{f(x_2) - f(x_1)}$
Find	$f(x_3)$. If $f(x_3) > 0$, then
Select interval	(x_1, x_3) If $f(x_1).f(x_3) < 0$
	(x_2, x_3) If $f(x_2).f(x_3) < 0$
Repeat above	For the new interval

EXAMPLE on FALSE POSITION

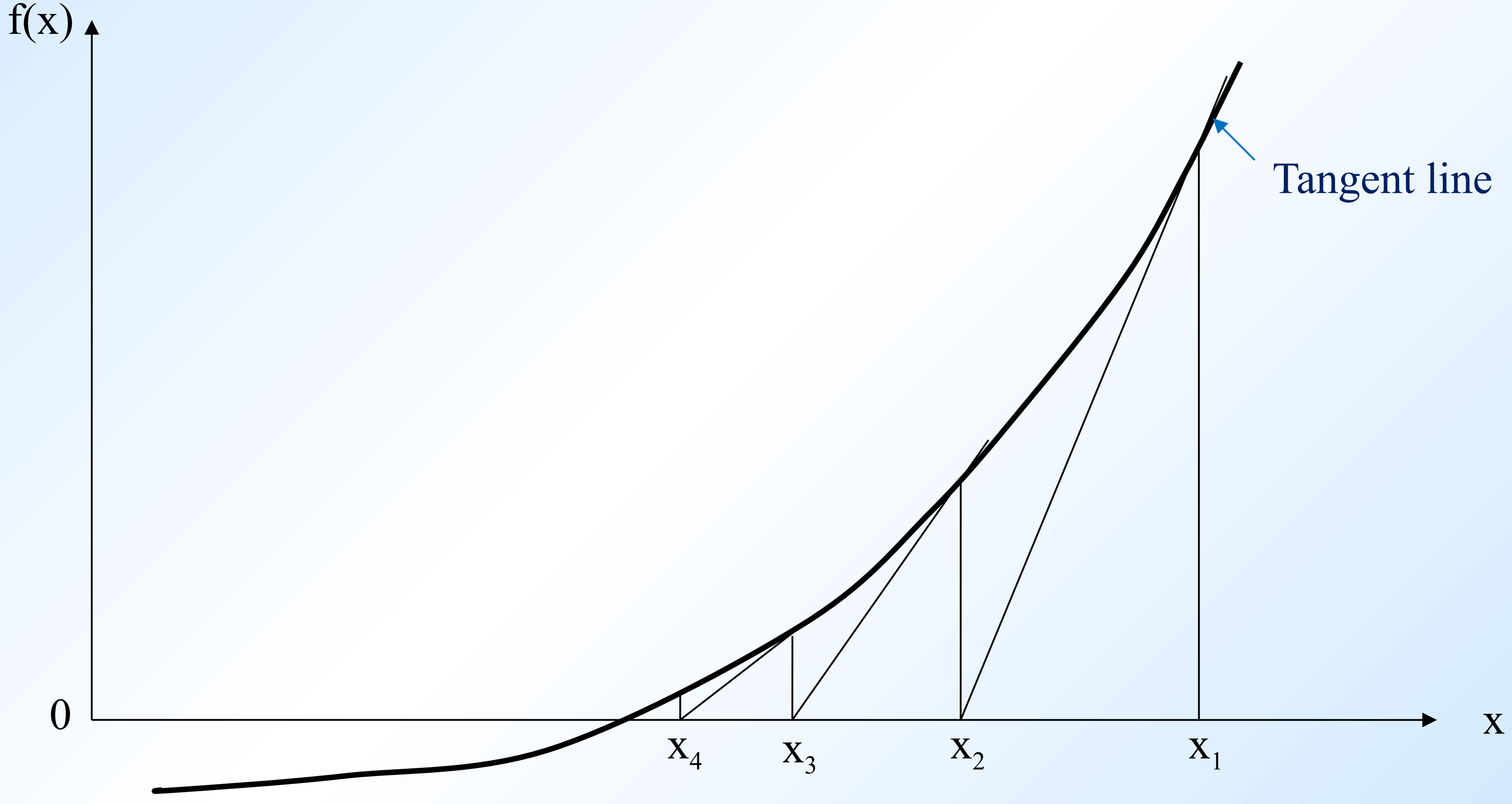
Find the root of $f(x) = x - x^{1/3} - 2 = 0$

in [3,4] with false position.

n	x_{left}	x_{right}	$x_{cross\ over}$	f(x)
1	3	4	3.5173426178	$-0.34555\ 10^{-2}$
2	3.5173426178	4	3.5213512486	$-0.24360\ 10^{-4}$
3	3.5213512486	4	3.5213795063	$-0.17160\ 10^{-6}$
4	3.5213795063	4	3.5213797054	$-0.12088\ 10^{-8}$

$$E_{est} = x_5 - x_4$$

NEWTON'S METHOD
(NEWTON-RAPHSON)



Condition: f(x) is continuous in (x_1, x_r) and f '(x) exists and does not change sign in (x_1, x_r)

Procedure:	Assume f(x) is linear over (x_1, x_r)	
	Guess x_1	
	Find x_2	$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$
	If f (x_2) not equal to zero, find x_3	$x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)}$
	Repeat until desired accuracy	$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

Recursive relation

EXAMPLE on NEWTON's METHOD

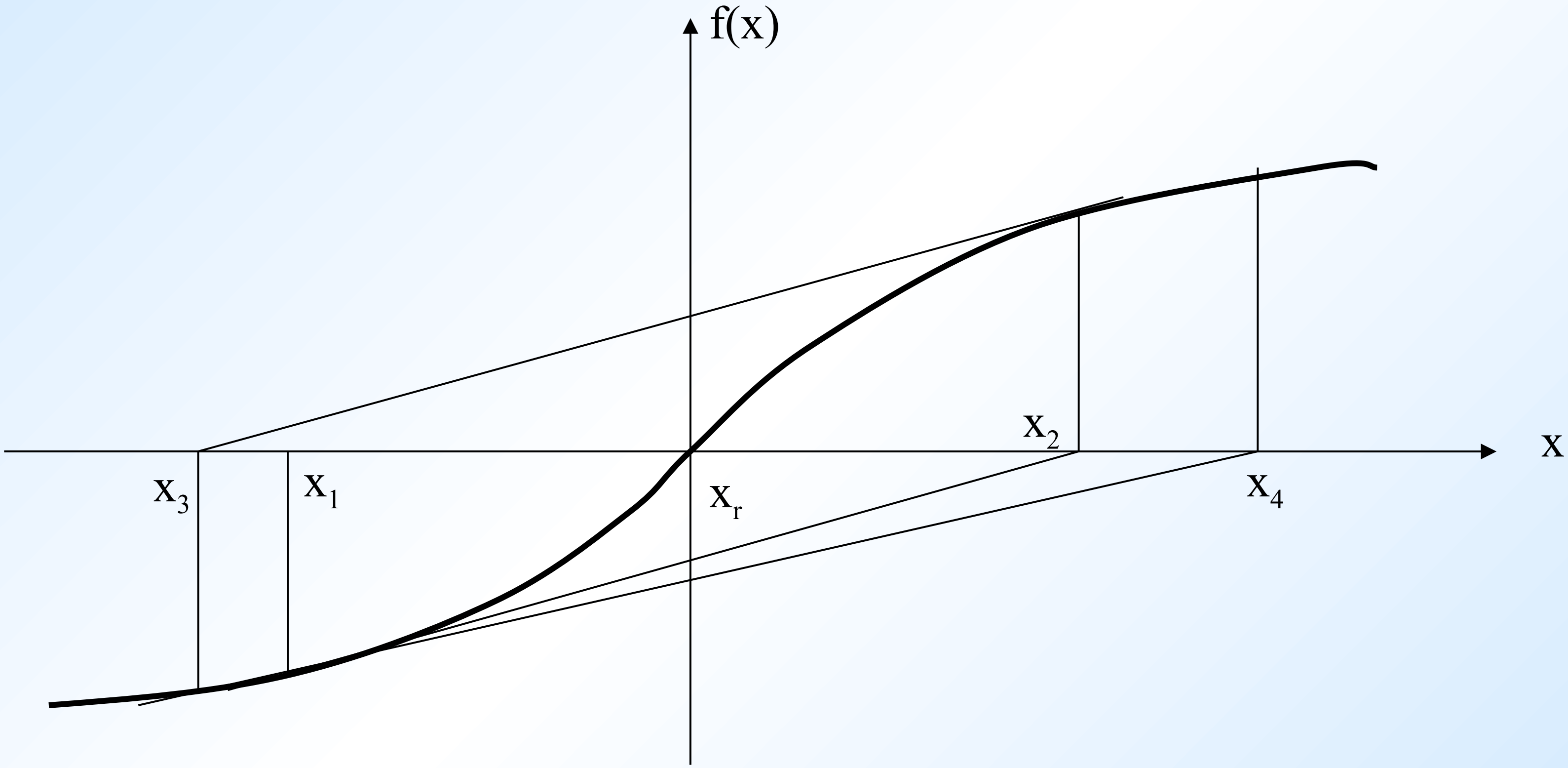
Find the root of $f(x) = x - x^{1/3} - 2 = 0$ in the range [3,4]

First derivative is	$f'(x) = 1 - (x^{-2/3})/3$
The iteration formula is	$x_{n+1} = x_n - \dfrac{x_n - x_n^{1/3} - 2}{1 - (x_n^{-2/3})/3}$

.

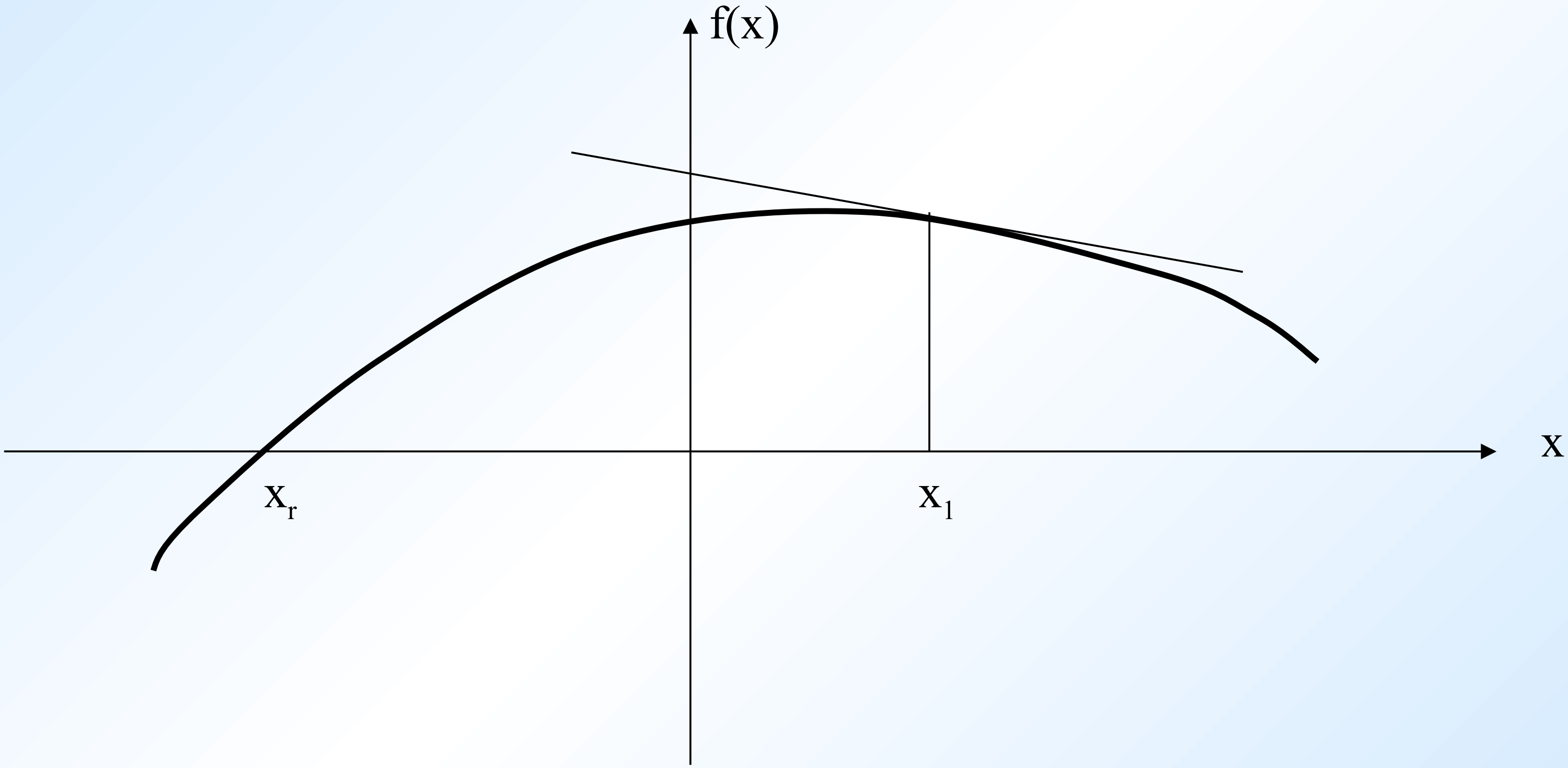
n	x_n	$f'(x_n)$	$f(x)$
0	3	0.83975005	- 0.44224957
1	3.52664429	0.85612976	0.00450679
2	3.52138015	0.85598641	$3.771 \; 10^{-7}$
3	3.52137971	0.85598640	$2.664 \; 10^{-15}$
4	3.52137971	0.85598640	0.0

Cases where NEWTON's METHOD do not work



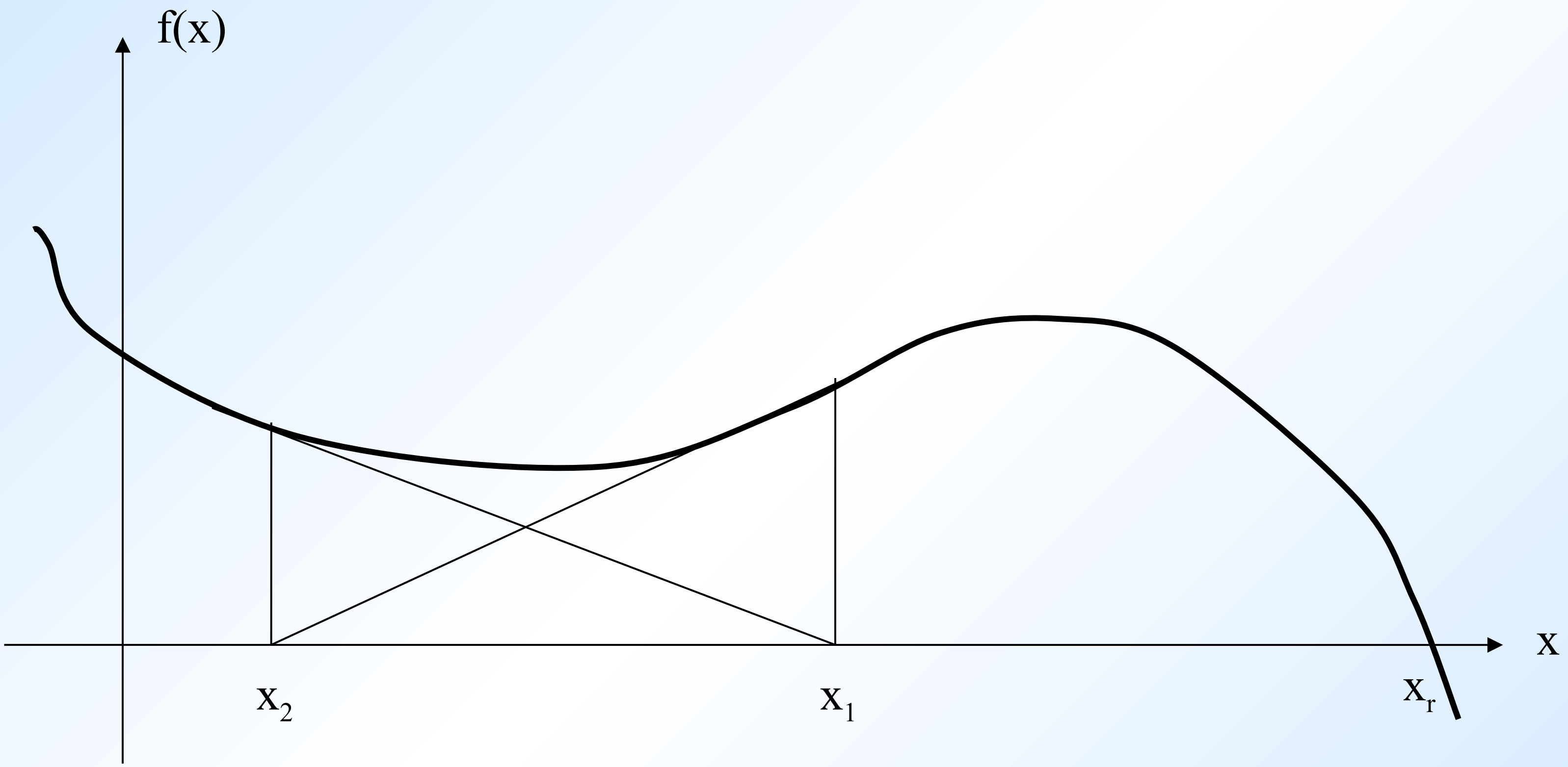
f ''(x) = 0 (Divergence)

Cases where NEWTON's METHOD do not work



Derivative (slope) at x_1 is close to zero

Cases where NEWTON's METHOD do not work



Derivative (slope) varies rapidly resulting in oscillations between x_1 and x_2

TAYLOR SERIES & NEWTON's METHOD

Drive the recursive relation for the Newton's method using Taylor series expansion

$$f(x) = f(x_0) + (x - x_0)\, f'(x_0) + \frac{(x - x_0)^2}{2!} f''(x_0) + \ldots$$

.

$$f(x_{n+1}) \cong f(x_n) + (x_{n+1} - x_n)\, f'(x_n) \cong 0$$

Solve for x_{n+1}:

$$x_{n+1} \cong x_n - \frac{f(x_n)}{f'(x_n)}$$

TAYLOR SERIES & NEWTON's METHOD

Show that Newton's method has quadratic convergence using Taylor series

$$f(x_{n+1}) \cong f(x_n) + (x_{n+1} - x_n) f'(x_n) \cong 0$$

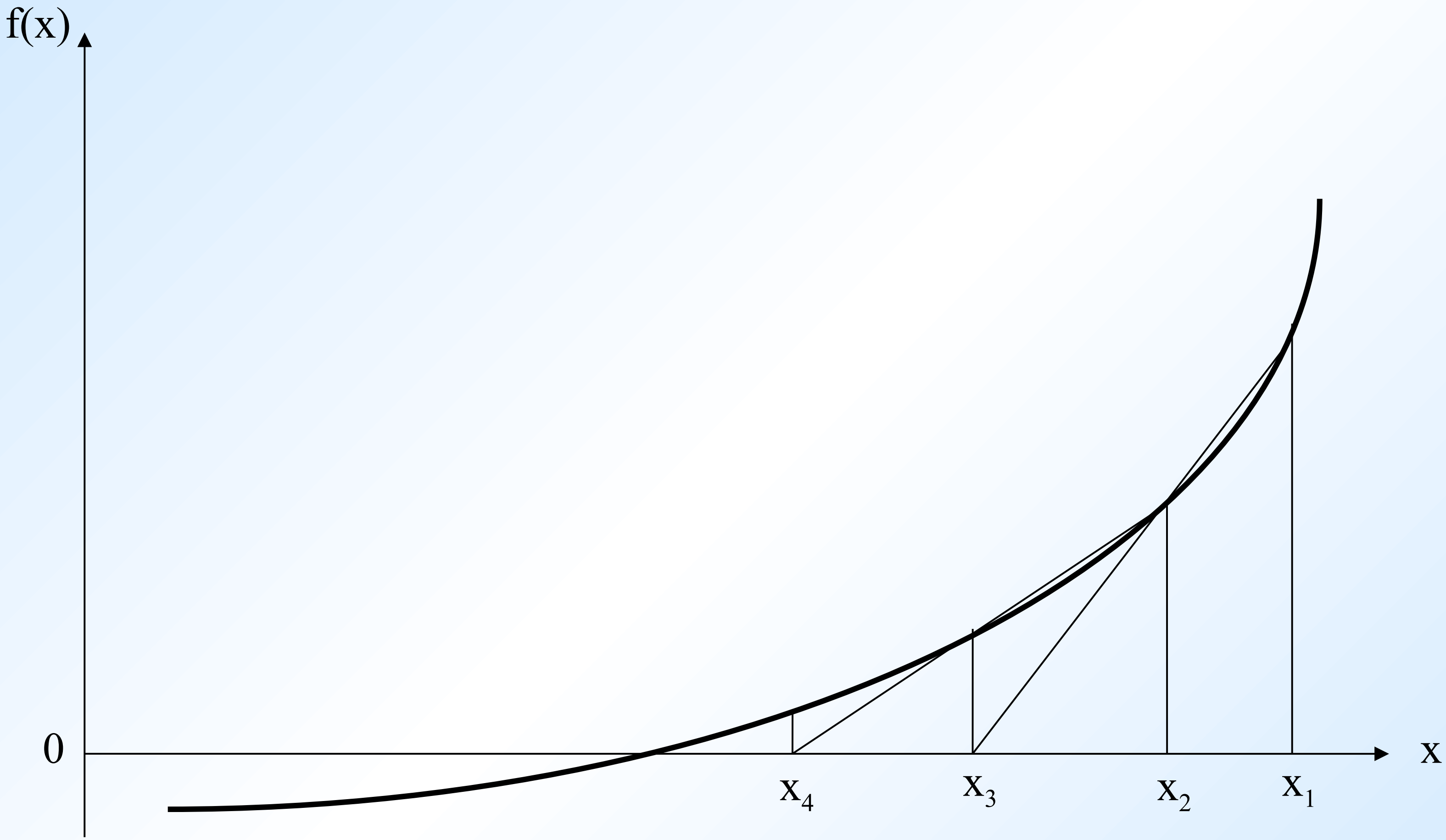
$$f(x_r) = f(x_n) + (x_r - x_n) f'(x_n) + \frac{(x_r - x_n)^2}{2} f''(\xi) \equiv 0$$

Substract: $$(x_r - x_{n+1}) f'(x_n) \cong \frac{-(x_r - x_n)^2}{2} f''(\xi) \equiv 0$$

Define: $E_{n+1} = x_r - x_{n+1}$ and $E_n = x_r - x_n$

Therefore: $E_{n+1} \propto E_n^2$

SECANT METHOD



Condition: Same as Newton Raphson except an implicit function for f (x) is not given. Replace the first derivative with

$$f'(x_n) \cong \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

Then

$$x_{n+1} = \frac{x_{n-1}\, f(x_n) - x_n\, f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

Procedure: Start with two initial guesses, x_0 and x_1

Find x_2 using the above expression

If $f(x_2) \neq 0$, repeat above until desired accuracy

EXAMPLE on SECANT

Find the root of $f(x) = x - x^{1/3} - 2 = 0$ in the range [3,4]

n	x_{n-1}	x_n	$f(x_n)$
0	4	3	- 0.44224957
1	3	3.51734262	- 0.00345547
2	3.51734262	3.52141665	0.00003163
3	3.52141665	3.52137970	$-2.034 \; 10^{-9}$
4	3.52137959	3.52137971	$-1.332 \; 10^{-15}$
5	3.52137971	3.52137971	0.0

FIXED-POINT ITERATION METHOD

Given: $f(x) = 0$ real-valued and continous

Procedure:	Re-write $f(x) = 0$ in the form $x = g(x)$
	Guess x_0
	If $x_0 \neq g(x_0)$ $\quad x_1 = g(x_0)$
	If $x_1 \neq g(x_1)$ $\quad x_2 = g(x_1)$
	Continue until desired accuracy

$$\left| x_{n+1} - x_n \right| \leq \varepsilon_1 \quad \text{and} \quad f(x_n) \leq \varepsilon_2$$

EXAMPLE on FIXED-POINT ITERATION

Find the root of $f(x) = x - x^{1/3} - 2 = 0$ in the range [3,4]

Rewrite as $x_{new} = g_1(x_{old}) = x_{old}^{1/3} + 2$ or

$$x_{new} = g_2(x_{old}) = (x_{old} - 2)^3 \quad \text{or}$$

$$x_{new} = g_3(x_{old}) = \frac{6 + 2\,x_{old}^{1/3}}{3 - x_{old}^{-2/3}}$$

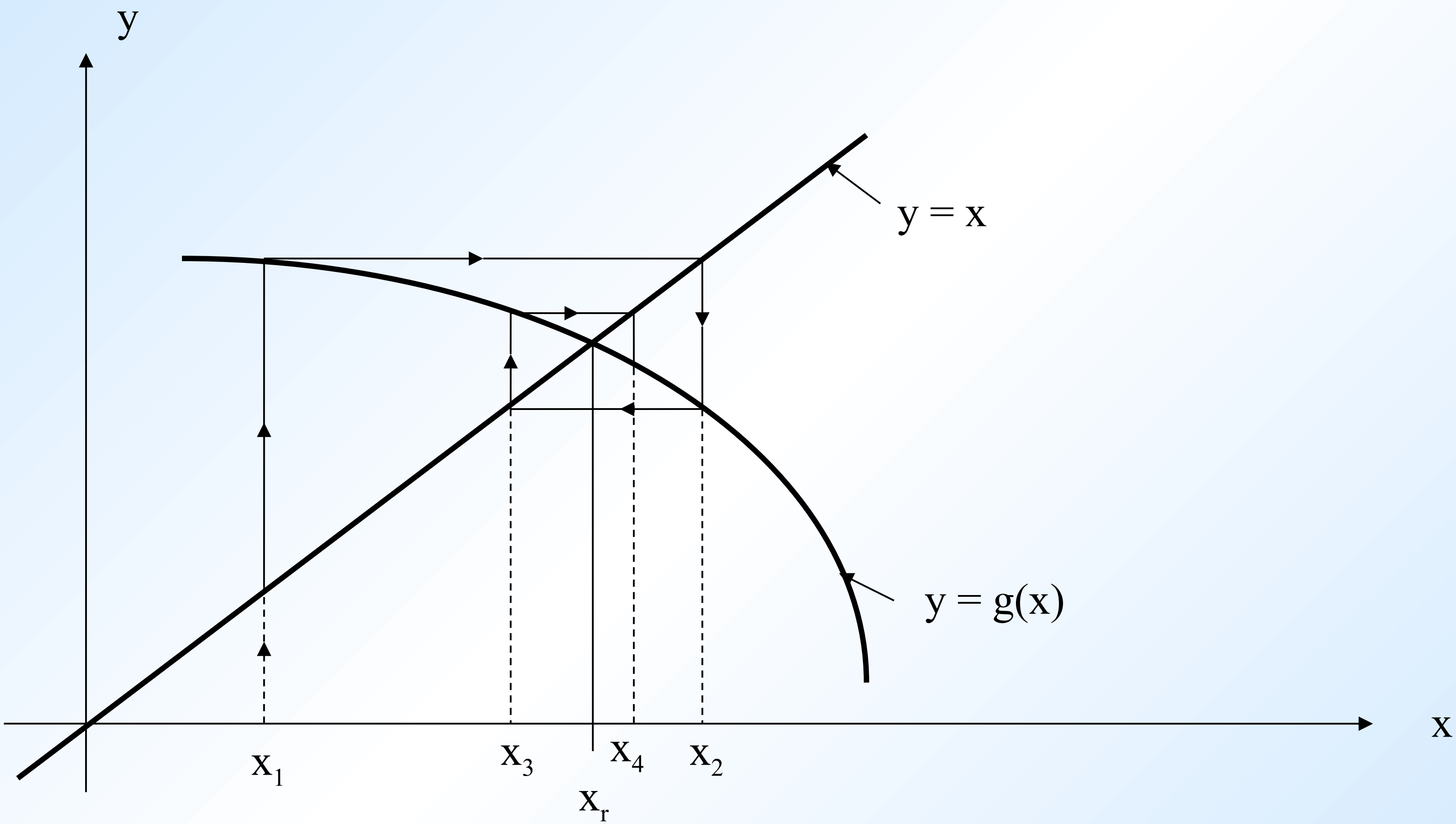
n	$g_1(x_{n-1})$		
0	3		
1	3.4422495703		
2	3.5098974493		
3	3.5197243050		
4	3.5211412691		
5	3.5213453678		
6	3.5213747615		
7	3.5213789946		
8	3.5213796042		
9	3.5213796920		

n	$g_1(x_{n-1})$	$g_2(x_{n-1})$	
0	3	3	
1	3.4422495703	1	
2	3.5098974493	- 1	
3	3.5197243050	- 27	
4	3.5211412691	- 24389	
5	3.5213453678	$-1.451\ 10^{13}$	
6	3.5213747615	$-3.055\ 10^{39}$	
7	3.5213789946	$-2.852\ 10^{118}$	
8	3.5213796042	∞	
9	3.5213796920	∞	

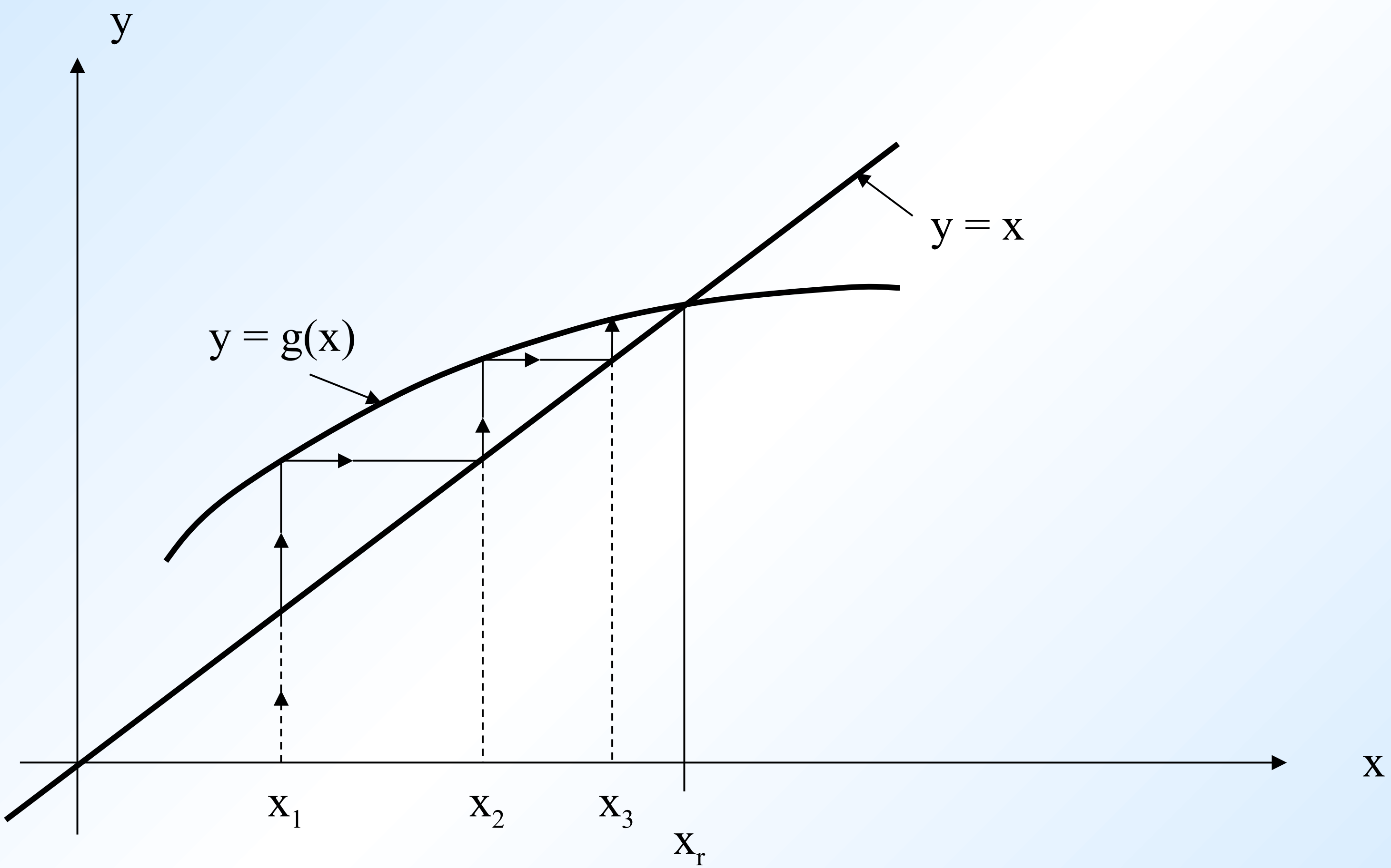
n	$g_1(x_{n-1})$	$g_2(x_{n-1})$	$g_3(x_{n-1})$
0	3	3	3
1	3.4422495703	1	3.5266442931
2	3.5098974493	- 1	3.5213801474
3	3.5197243050	- 27	3.5213797068
4	3.5211412691	- 24389	3.5213797068
5	3.5213453678	$-1.451 \; 10^{13}$	3.5213797068
6	3.5213747615	$-3.055 \; 10^{39}$	3.5213797068
7	3.5213789946	$-2.852 \; 10^{118}$	3.5213797068
8	3.5213796042	∞	3.5213797068
9	3.5213796920	∞	3.5213797068

Convergence Criteria for Fixed-Point Iteration

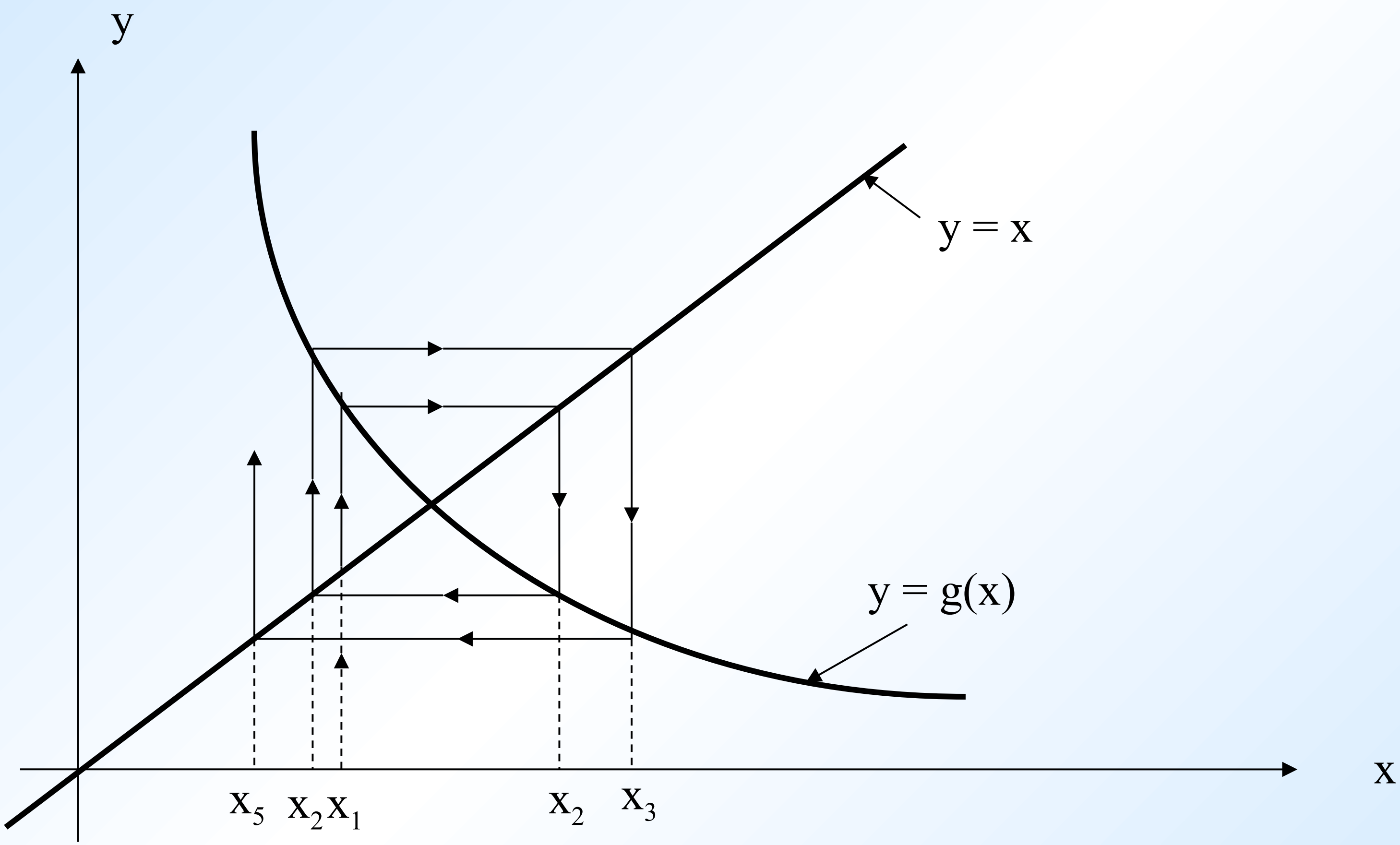
- There is an interval [a , b] such that for all x in [a , b] g(x) is defined and is in [a, b], i.e. g(x) maps [a , b] onto itself;
- g(x) is continuous on [a , b];
- g(x) is differentiable on [a , b];
- There exists a non-negative constant K < 1 such that, for all x in [a , b], $g'(x) \leq K$. In other words, $-1 \leq g'(x) \leq +1$



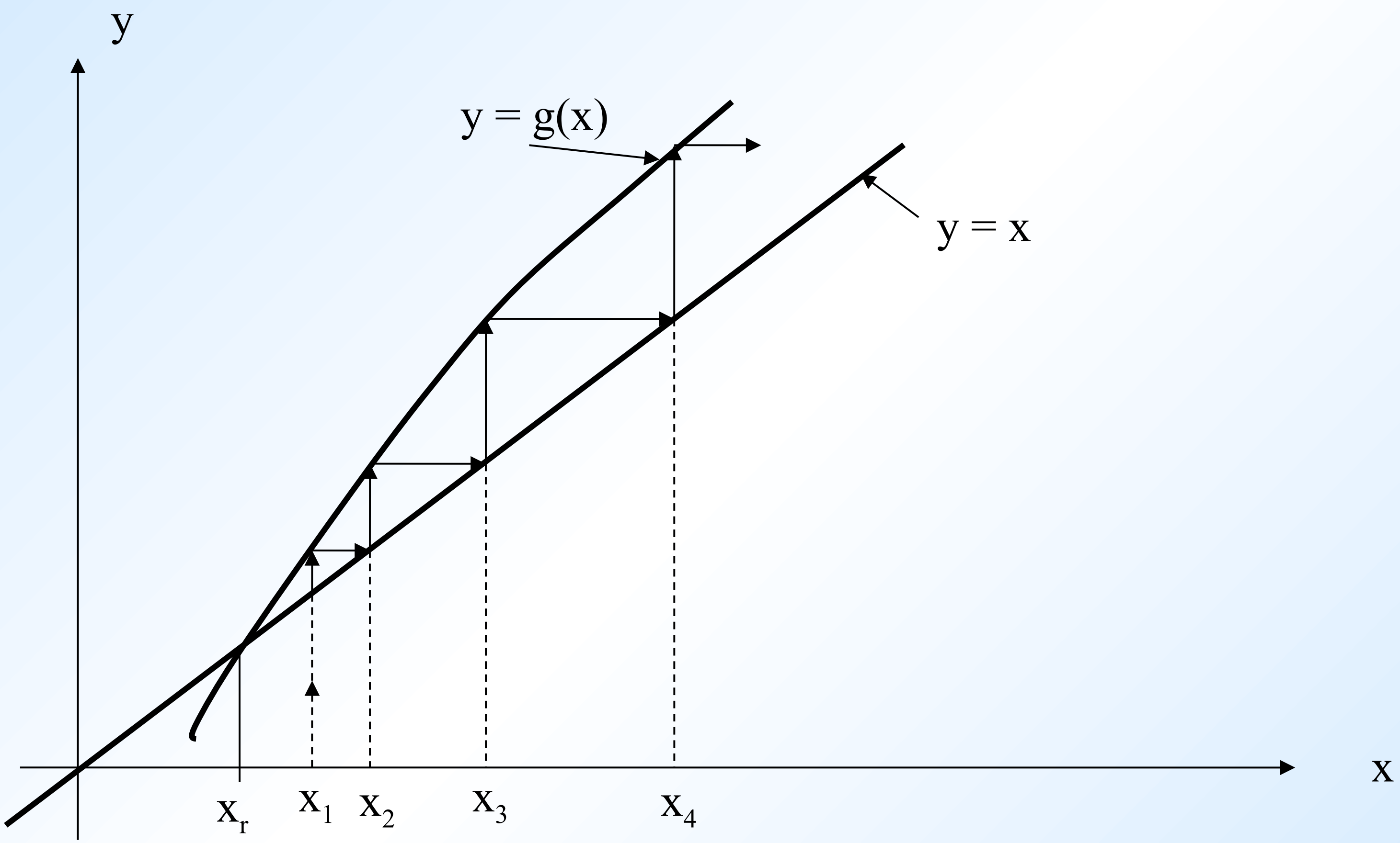
$-1 < g'(x) < 0$ Convergent



$0 < g'(x) < 1$ Convergent



$g'(x) < -1$ Divergent



g ‘(x) > 1 Divergent

Multiple Roots

Multiple roots (of multiplicity p) occurs when

$f(x_r) = f'(x_r) = \dots = f^{(p-1)}(x_r) = 0$ and $f^{p}(x_r) \neq 0$

or

$f(x) = (x - x_r)^p\, \varphi(x)$ where $\varphi(x) \neq 0$ near x_r

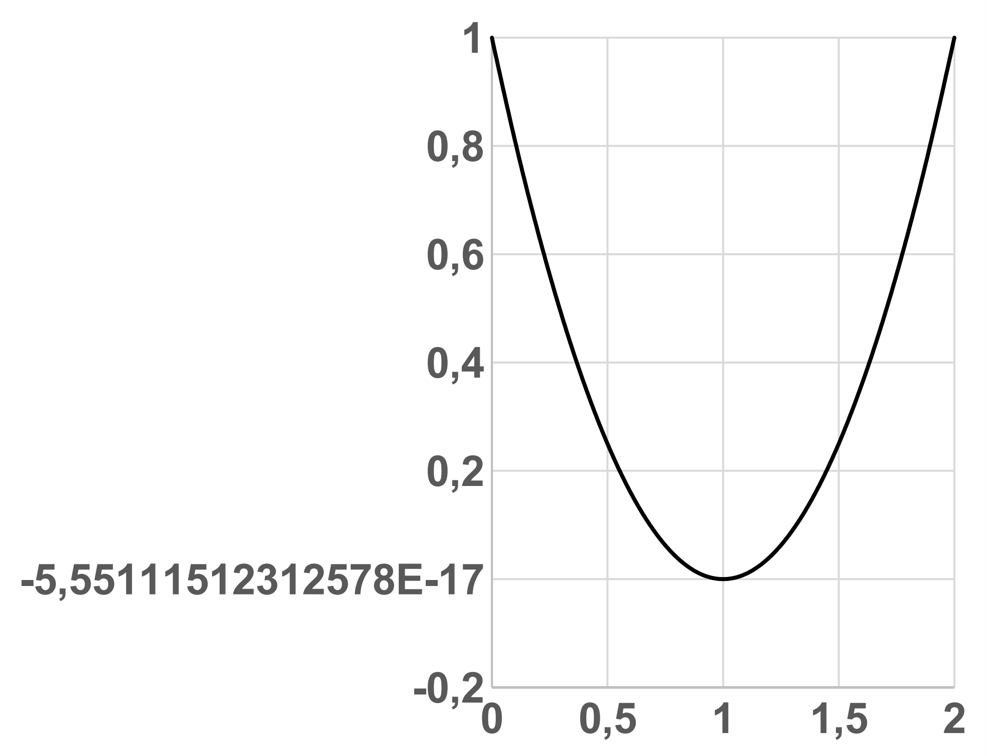
Consider $g(x) = \dfrac{f(x)}{f'(x)} = 0$ and find the root of g(x) instead of f(x)

$$g(x) = \frac{f(x)}{f'(x)} = \frac{(x - x_r)\ \varphi(x)}{p\ \varphi(x) + (x - x_r)\ \varphi'(x)} = 0$$

Show that $g'(x_r) \approx \dfrac{1}{p}$

Example

Find the roots of the equation $f(x) = x^2 - 2x + 1 = 0$

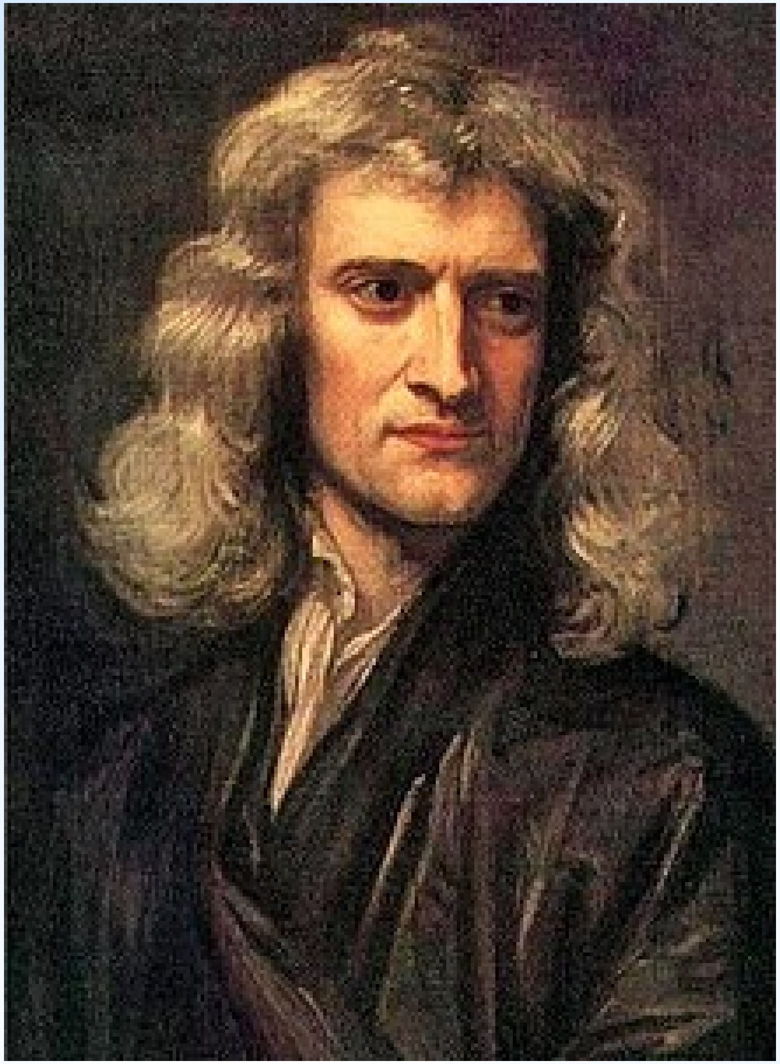


Newton-Raphson on f(x)			
n	x_n	$f(x_n)$	$f'(x_n)$
0	1.1000000	0.01	0.2
1	1.0500000	0.0025	0.1
2	1.0250000	0.000625	0.05
3	1.0125000	0.00015625	0.025
4	1.0062500	3.9062E-05	0.0125
5	1.0031250	9.7656E-06	0.00625
6	1.0015625	2.4414E-06	0.003125
7	1.0007812	6.1035E-07	0.0015625
8	1.0003906	1.5259E-07	0.00078125
9	1.0001953	3.8147E-08	0.00039063
10	1.0000977	9.5367E-09	0.00019531

Bernard Bolzano

Bohemian mathematician

1781 - 1848



Isaac Newton

English Mathematician

1642 - 1727

Isaac Newton (1642-1727) is generally regarded as one of the greatest mathematicians of all time. He entered Trinity College, Cambridge, in 1661 and graduated with a B.A. degree in 1665. In 1668, he received a master's degree, and was appointed Lucasian Professor of Mathematics, one of the most prestigious positions in English academia at the time. In his later years, Newton served in Parliament and was Warden of the Mint. In 1703, he was elected president of the Royal Society of London, of which he had been a member since 1672. Two years later he was knighted by Queen Anne.

Newton is given co-credit, along with the German Wilhelm Gottfried von Leibniz, for the discovery and development of the calculus, work that Newton did in the period 1664-1666, but did not publish until years later, thus laying the groundwork for an ugly argument with Leibniz over who should get credit for the discovery. In 1687, at the urging of the astronomer Edmund Halley, Newton published his ground-breaking compilation of mathematics and science, Principia Mathematica, which is apparently the first place that the root-finding method that bears his name appears, although he probably had used it as early as 1669.