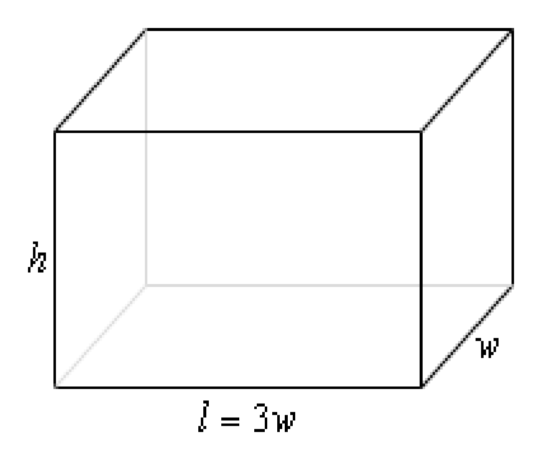
OPTIMIZATION

Problems with an objective function and constrain(s)

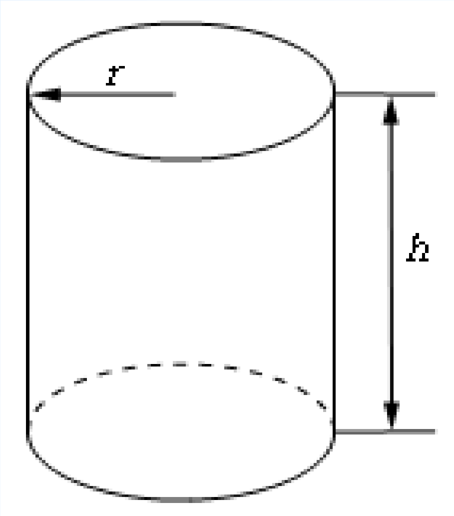
Example 1: We want to construct a box whose base length is 3 times the base width. The material used to build the top and bottom cost 100 TL/m^2 and the material used to build the sides cost 60 TL/m^2. If the box must have a volume of 5 m^3, determine the dimensions that will minimize the cost to build the box.



Minimize: $C = 100\,(2\,l\,w) + 60\,(2\,w\,h + 2\,l\,h)$
$= 600\,w^2 + 480\,w\,h$

Constraint: $5 = l\,w\,h = 3\,w^2\,h$

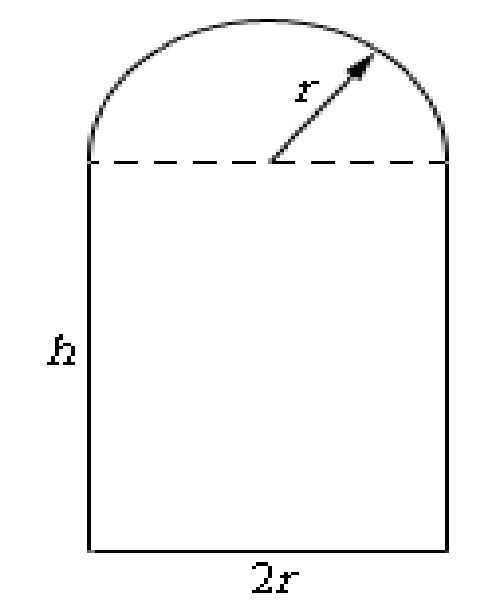
Example 2: A manufacturer needs to make a cylindrical can that will hold 1.5 liters of liquid. Determine the dimensions of the can that will minimize the amount of material used in its construction.



Minimize: $A = 2\pi r h + 2\pi r^2$ in cm^2

Constraint: $1500 = \pi r^2 h$ in cm^3

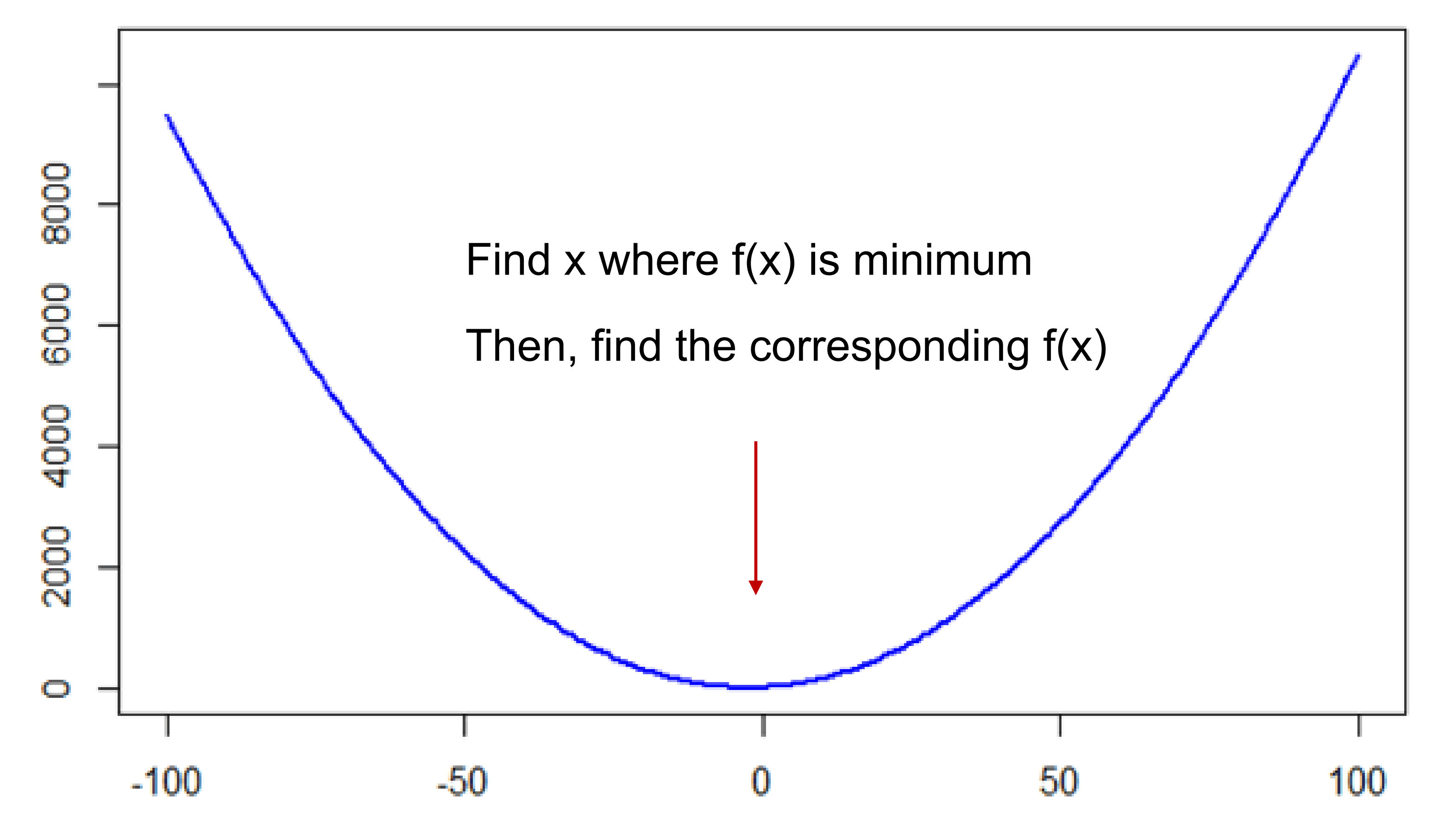
Example 3: A window is being built and the bottom is a rectangle and the top is a semicircle. If there is 12 meters of framing material, what must the dimensions of the window be to let in the most light?



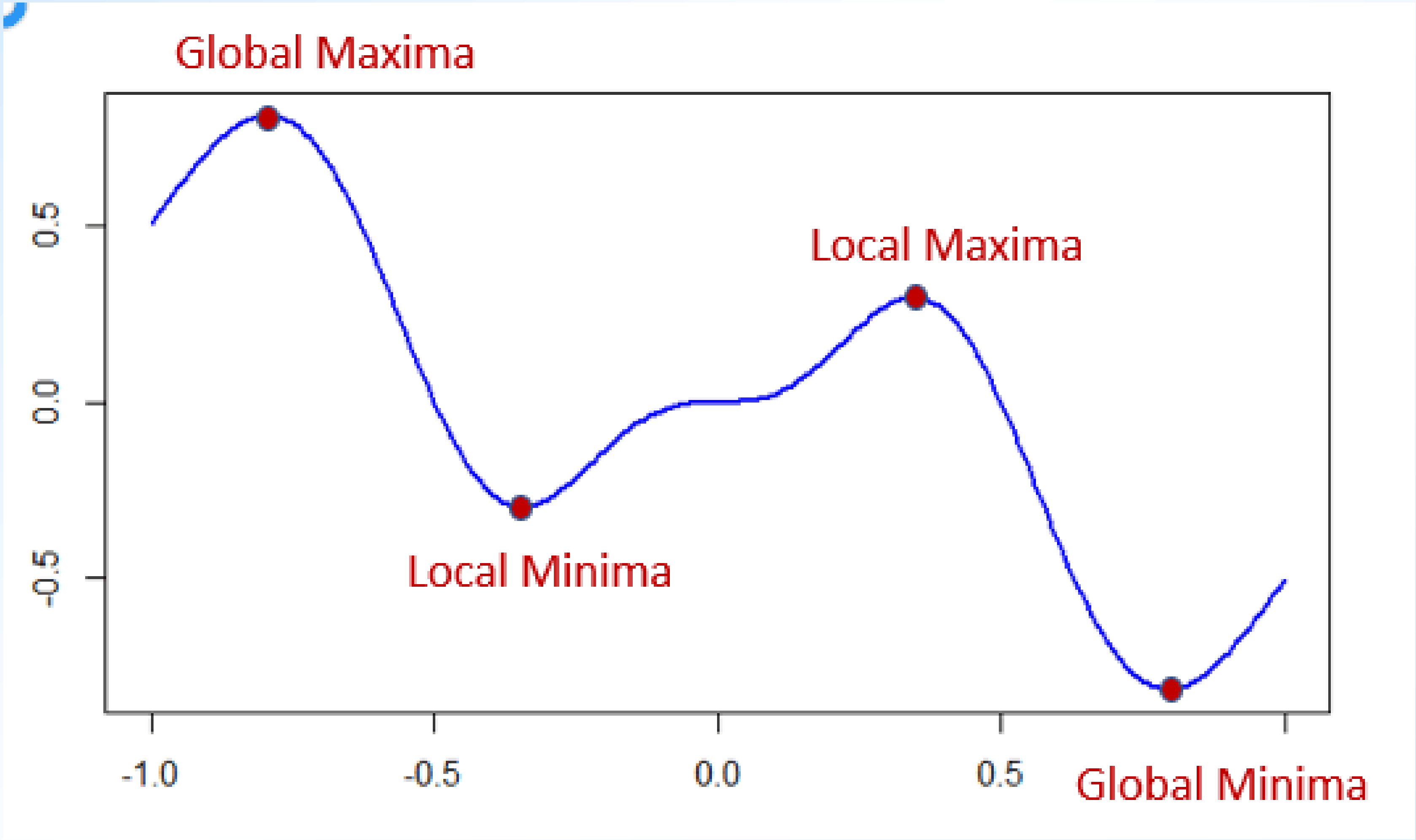
Maximize: $A = 2\,h\,r + \frac{1}{2}\,\pi\,r^2$

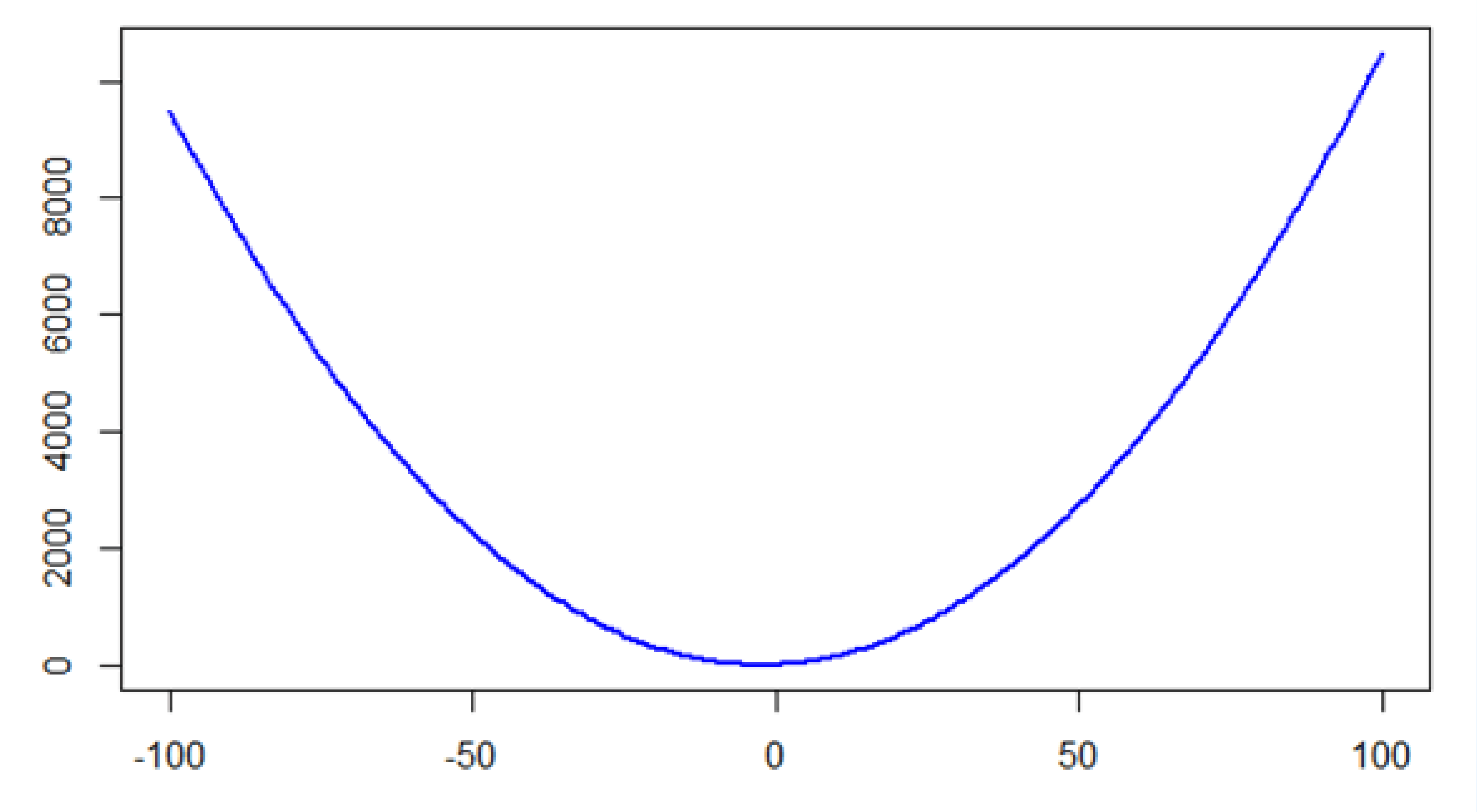
Constraint: $12 = 2\,h + 2\,r + \pi\,r$

Problem: Given a function, f(x), unimodal in a certain range, find the minimum (or the maximum) in this range



Non-Unimodal function



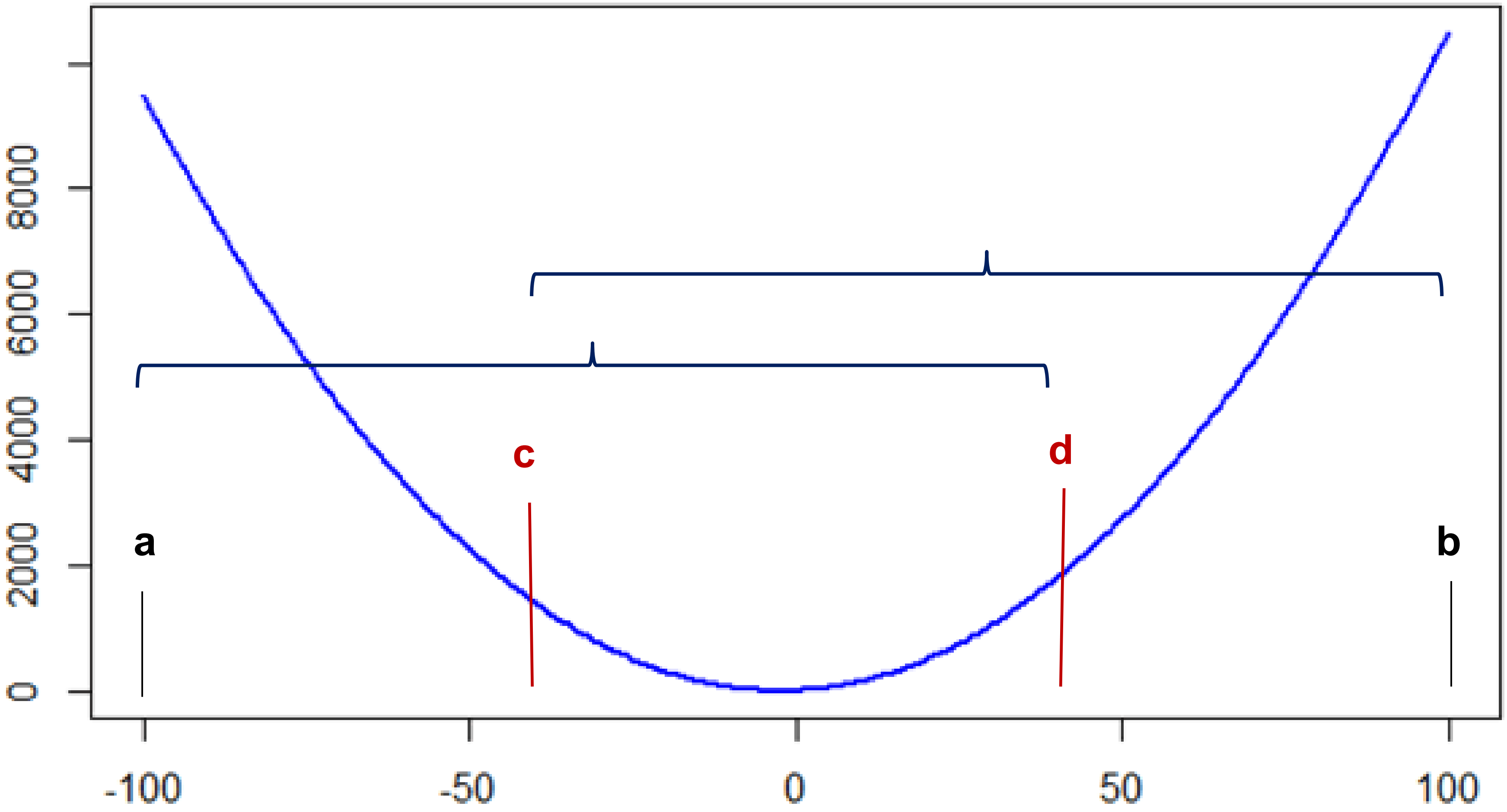


a b

f(x) is unimodal in (a,b), i.e., only one minimum or maximum

Procedure

- Choose two points, c and d, in (a,b)
- Decide in which bracket (range) the minimum (or maximum) lies, (a,d) or (c,b)
- Choose that smaller bracket and update a and b
- Repeat above and home in to the minimum or maximum


8000
6000
4000
2000
0
a
b
c
d
-100
-50
0
50
100

Question:

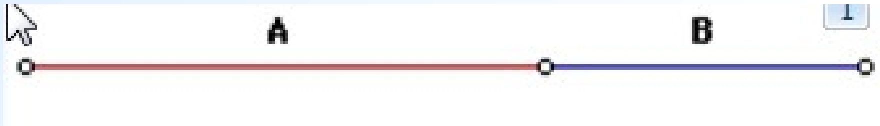
- How do you choose the points, c and d?

Answer:

- Anyway you like.

In order to save time, you may use **golden ratio**.

Golden Section



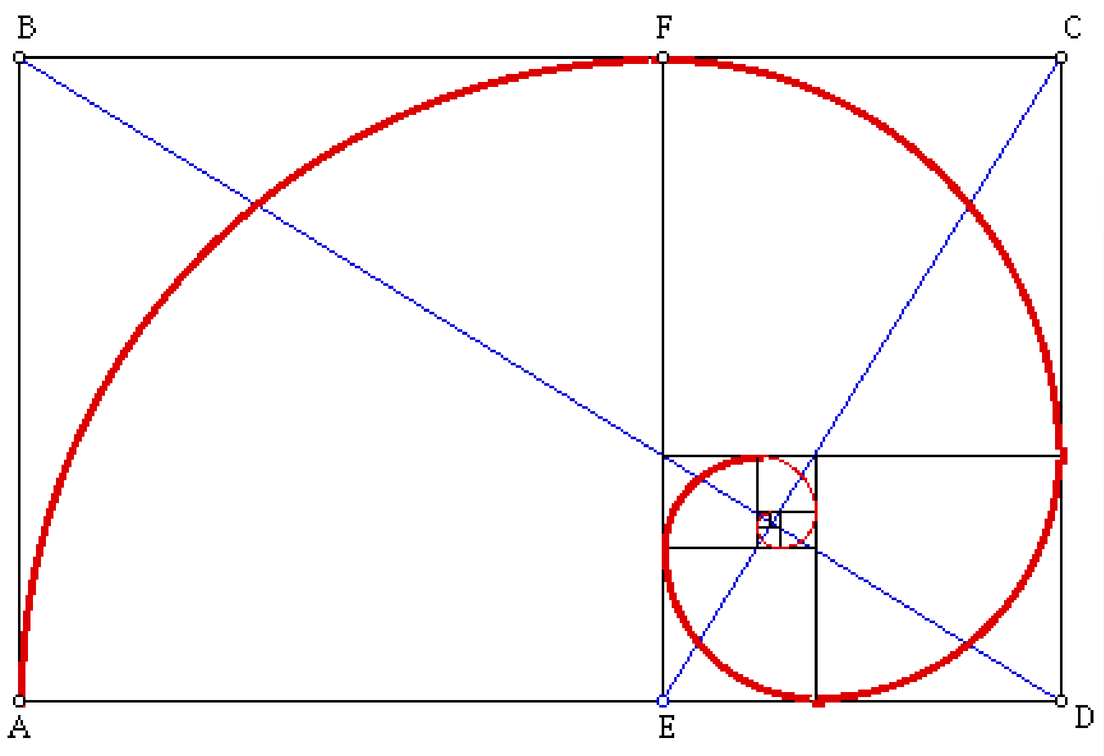
The golden section (or ratio) is a certain length that is divided in such a way that the ratio of the longer part to the whole is the same as the ratio of the shorter part to the longer part.

$$r = \frac{A}{A+B} = \frac{B}{A} = \frac{\sqrt{5} - 1}{2} = 0.618034$$

r is the positive root of equation $r^2 + r - 1 = 0$

$$\varphi = \frac{A+B}{A} = \frac{A}{B} = \frac{\sqrt{5} + 1}{2} = 1.618034$$

Golden Rectangle:



Read: https://en.wikipedia.org/wiki/Golden_ratio

Golden Ratio (Section) Search

Definitions: Golden ratio is: $r = \frac{\sqrt{5} - 1}{2} = 0.61803398874989$

Unimodal Function

A function f (x) is unimodal on [a,b] if there exist a unique number p in [a,b] such that

- f(x) is decreasing on [a,p] and increasing on [p,b] (for a minimum) or
- f(x) is inreasing on [a,p] and decreasing on [p,b] (for a maximum)

If f (x) is known to be unimodal on [a,b], it is possible to replace the interval with a subinterval on which f (x) takes on its minimum (or maximum) value.

The golden search requires that two interior points

$$c = a + (1 - r)(b - a)$$

and

$$d = a + r(b - a)$$

be used where r is the golden ratio. This results in

$$a < c < d < b.$$

The condition that $f(x)$ is unimodal guarantees that the functional values, $f(c)$ and $f(d)$, are less than $\max[f(a), f(b)]$ when there is a minimum, or more than $\min[f(a), f(b)]$ when there is a maximum.

For the case when there is a **minimum**:

If $f(c) \le f(d)$, then the minimum must occur in the subinterval [a,d] and we replace b with d and continue the search in the new subinterval.

If $f(d) < f(c)$, then the minimum must occur in [c,b] and we replace a with c and continue the search.

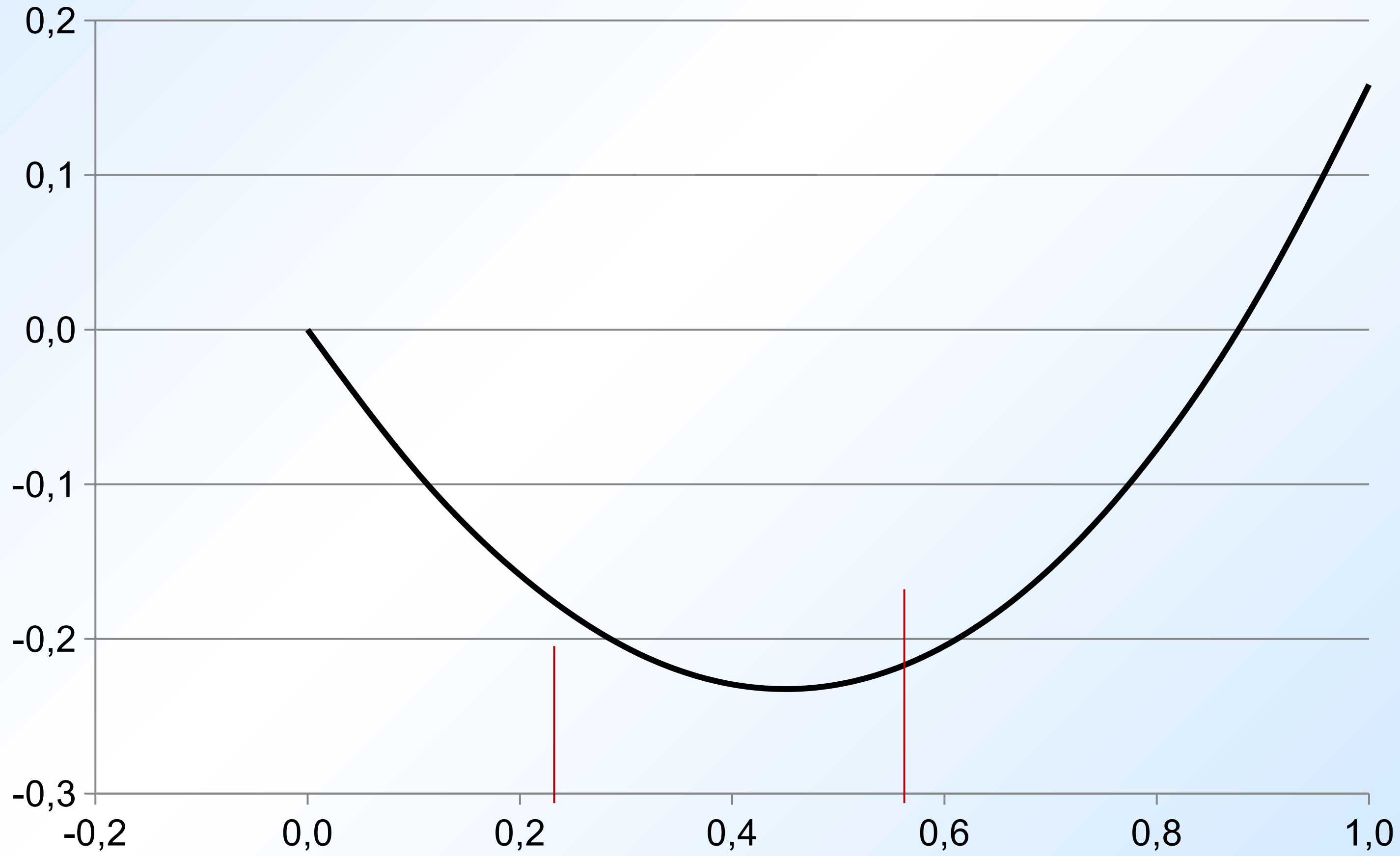
For the case when there is a **maximum**:

If $f(c) \ge f(d)$, then the maximum must occur in the subinterval [a,d] and we replace b with d and continue the search in the new subinterval.

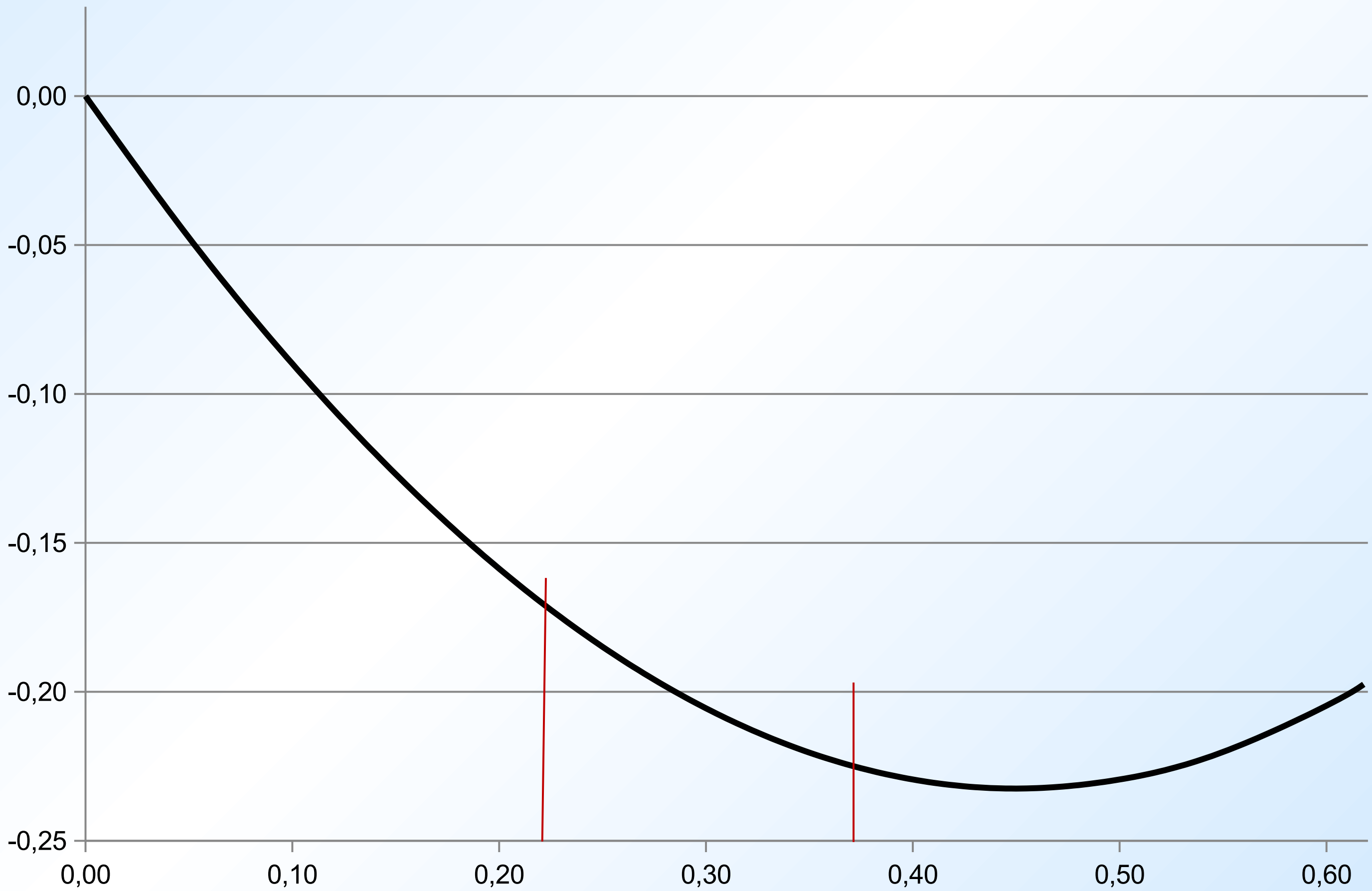
If $f(d) > f(c)$, then the maximum must occur in [c,b] and we replace a with c and continue the search.

Example on Golden Ratio Search

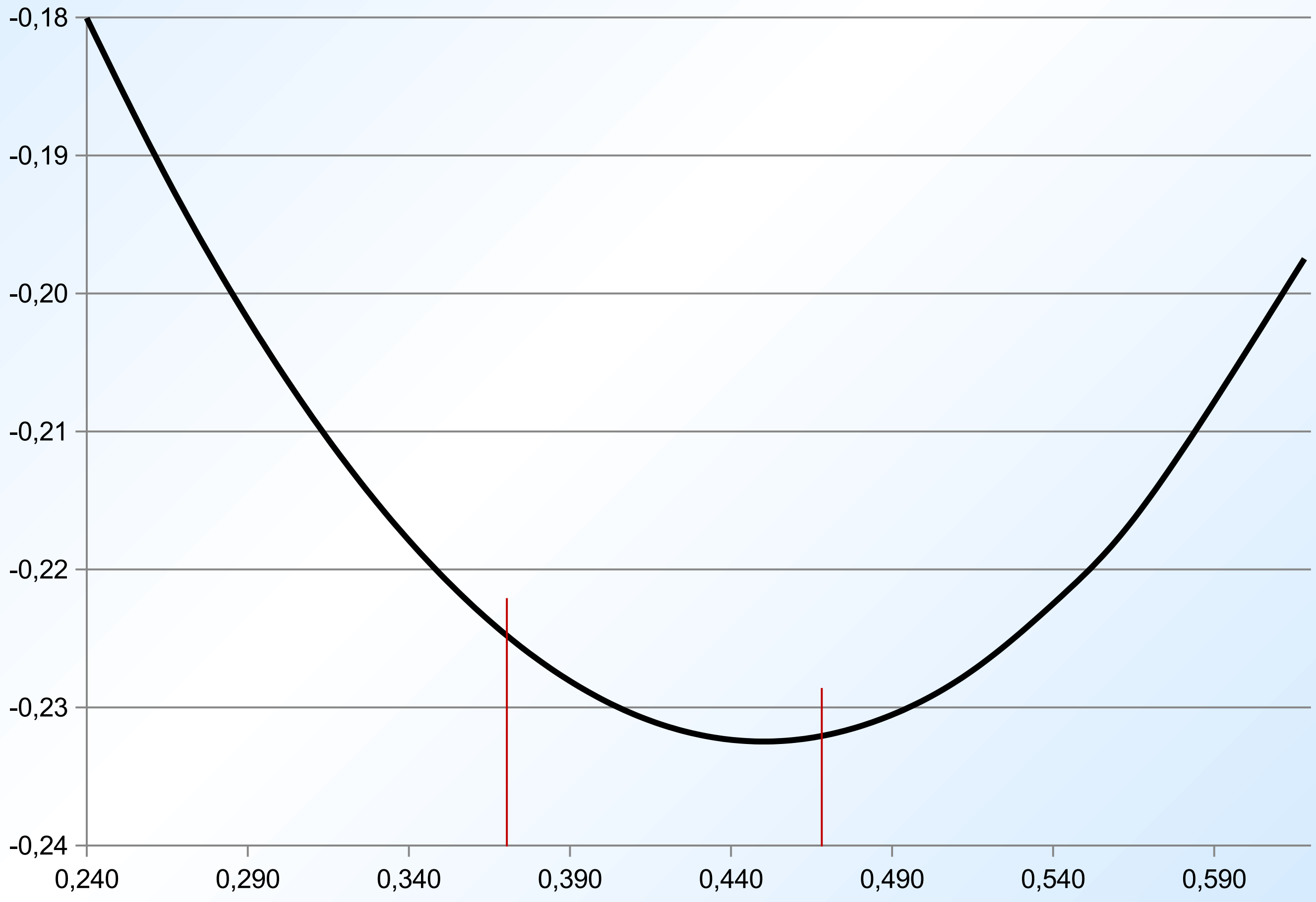
Find the minimum of the unimodal function $f(x) = x^2 - \sin(x)$ in $[0,1]$ using golden section search.



The search range is reduced to [0 , 0.618]:



The search range is reduced to [0.236 , 0.618]:



The complete solution:

k	a_k	c_k	d_k	b_k	$f(c_k)$	$f(d_k)$
0	0.0000000	0.3819660	**0.6180340**	1.0000000	- 0.22684748	- 0.19746793
1	0.0000000	**0.2360680**	0.3819660	0.6180340	- 0.17815339	- 0.22684748
2	0.2360680	**0.3819660**	0.4721360	0.6180340	- 0.22684748	- 0.23187724
3	0.38195660	0.4721360	**0.5278640**	0.6180340	- 0.23187724	- 0.22504882
4	0.38195660	0.4376941	**0.4721360**	0.5278640	- 0.23227594	- 0.23187724
5	0.38195660	**0.4164079**	0.4376941	0.4721360	- 0.23108238	- 0.23227594
6	0.4164079	**0.4376941**	0.4508497	0.4721360	- 0.23227594	- 0.23246503
.						
.						
21	0.4501574	**0.4501730**	0.4501827	0.4501983	- 0.23246558	- 0.23246558
22	0.4501730	**0.4501827**	0.4501886	0.4501983	- 0.23246558	- 0.23246558
23	0.4501827	**0.4501886**	0.4501923	0.4501983	- 0.23246558	- 0.23246558

Fibonacci Numbers

Leonardo of Pisa, better known as **Fibonacci**, was born in Pisa, Italy, about 1175 AD. He was known as the greatest mathematician of the middle ages. Completed in 1202, Fibonacci wrote a book titled ***Liber abaci*** on how to do arithmetic in the decimal system. Although it was Fibonacci himself that discovered the sequence of numbers, it was French mathematician, Edouard Lucas, who gave the actual name of "**Fibonacci numbers**" to the series of numbers that was first mentioned by Fibonacci in his book. Since this discovery, it has been shown that Fibonacci numbers can be seen in a variety of things.

Fibonacci (Leonardo of Pisa)

Italian Mathematician

1170 – 1250

Leonardo Pisano (1170-ca. 1250) is more widely known by his nickname, **Fibonacci**. He traveled widely in the Mediterranean region with his father, a diplomat in the service of Pisa. Fibonacci wrote a number of texts, the most famous of which is probably Liber abbaci, which appeared in 1202. This contains the famous problem that leads to the Fibonacci sequence:

A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?

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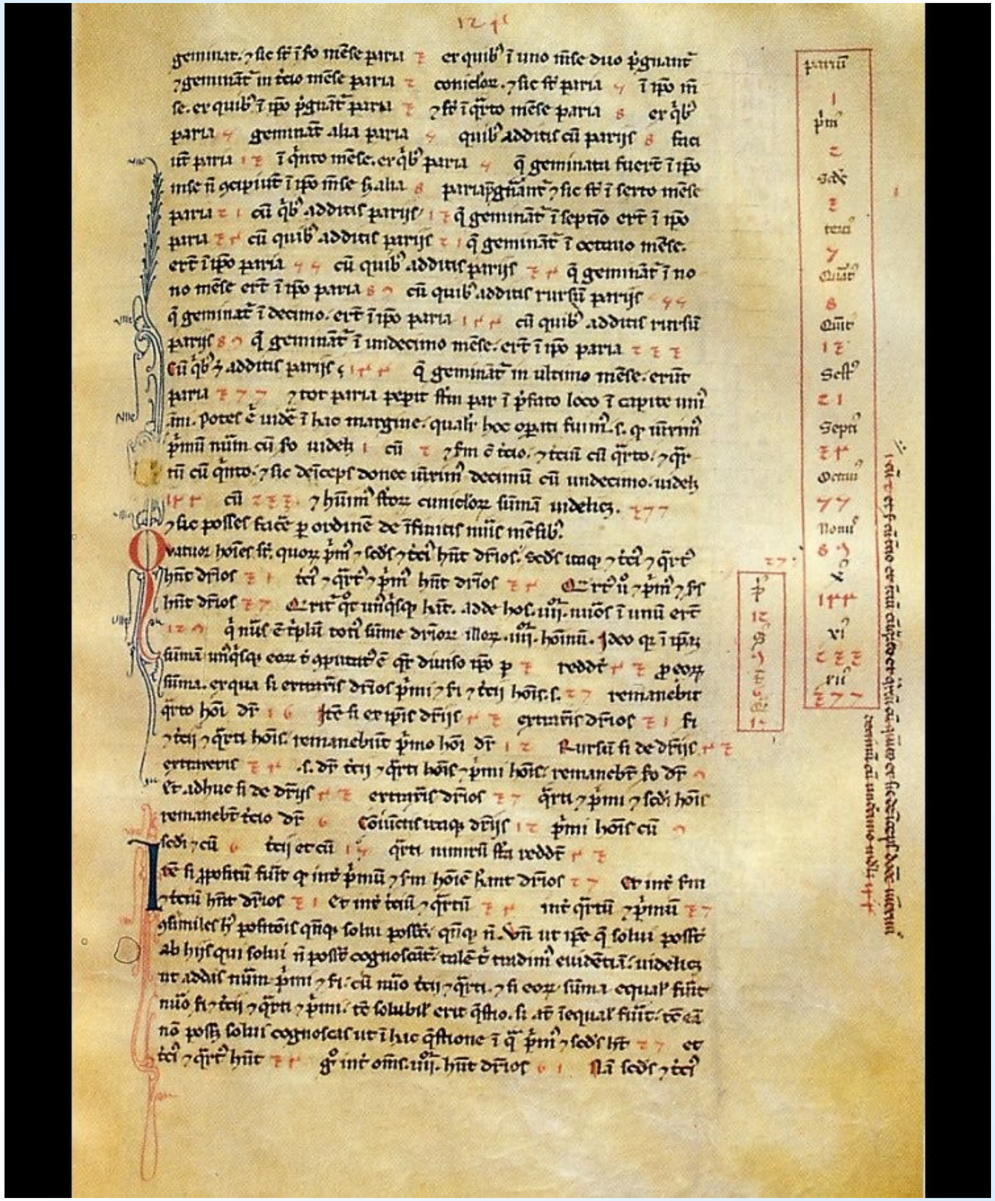

By charting the population of rabbits, Fibonacci discovered a number series from which one can derive the Golden Mean. The beginning of the sequence: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, etc ... Each number is the sum of the two preceding numbers.

Dividing each number in the series by the one which precedes, it produces a ratio which stabilizes around 1.618034

Explicit formula for n^{th} Fibonacci number:

$$F(n) = \frac{(\varphi)^n - \left(-\frac{1}{\varphi}\right)^n}{\sqrt{5}} \qquad \text{where} \qquad \varphi = \frac{1+\sqrt{5}}{\sqrt{5}}$$

If you count the number of petals in most flowers, you will find that the answer is a Fibonacci number. For example, an iris (süsen çiçeği) has 3 petals, a primrose (çuha çiçeği) 5, a delphinium (hezaren çiçeği) 8, ragwort 13, an aster (dalya) 21, daisies (papatya) 13, 21, or 34, and Michaelmas daisies 55 or 89 petals. All Fibonacci numbers.

If you look at a sunflower (ayçiçeği), you will see a beautiful pattern of two spirals, one running clockwise, the other counterclockwise. Count those spirals, and for most sunflowers you will find that there are 21 or 34 running clockwise and 34 or 55 counterclockwise, respectively - all Fibonacci numbers. Less common are sunflowers with 55 and 89, with 89 and 144, and even 144 and 233 in one confirmed case.

Other flowers exhibit the same phenomenon; the wildflower Black-Eyed Susan is a good example.

Similarly, pine cones have 5 clockwise spirals and 8 counterclockwise spirals, and the pineapple has 8 clockwise spirals and 13 going counterclockwise.

Read: http://www.maa.org/devlin/devlin_06_04.html

Read: «The Beauty of Numbers», a pdf article on the Moodle