CLASSIFICATION OF DIFFERENTIAL EQUATIONS

Differential Equations	Ordinary	ODE
	Partial	PDE

ODE	Linear	$y' + A x^2 y = f(x)$
	Non-linear	$(y')^2 + A y y' = f(x)$

ODE	Homogeneous	..... $= 0$
	Non-homogeneous	..... $= f(x) \neq 0$

ODE	Homogeneous	..... = 0
	Non-homogeneous	..... = f (x) $\neq$ 0

System of ODE's	y ' = f (x, y(x), z(x)) z ' = g (x, y(x), z(x))

ODE	Initial-value Problem	depending on the conditions
	Boundary-value Problem	

Physical Problems	Propagation Problem	open domain
	Equilibrium Problem	closed domain
	Eigenvalue Problem	

Solution Methods	Marching Numerical Method
	Equilibrium Numerical Method

LIPSHITZ CONDITION

Theorem: on existence of a solution for first-order ODE:

$$\frac{d\,y(t)}{d\,t} = f\,(t, y)$$

If f (t,y) is continuous on $a \le t \le b$, and there exists a constant L such that

$$\left| f(t,y) - f(t,z) \right| \le L \left| y - z \right|$$

for all $t \in [a , b]$, and all real y and z, then the initial-value problem

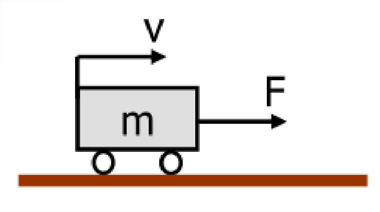
$$\frac{d\,y(t)}{d\,t} = f\,(t, y) \qquad \text{with} \qquad y(t_0) = y_0$$

where $t_0 \in [a , b]$ possesses a unique solution. This is called Lipschitz condition.

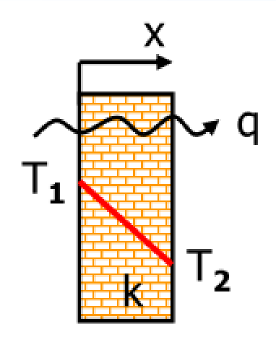
Rudolf Otto Sigismund Lipschitz (1832-1903) was born in Königsberg, Germany. He entered the University of Königsberg at the age of 15, and completed his studies at the University of Berlin, from which he was awarded a doctorate in 1853. By 1864, he was a full professor at the University of Bonn, where he remained the rest of his professional life. The Lipschitz condition first appeared in a paper on the existence of solutions to differential equations, which was published in book form as part of an 1877 treatise on analysis.

Examples of ODE's

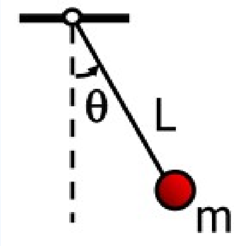
Newton's 2nd law of motion $F = m \dfrac{dv}{dt}$



Fourier's law of heat conduction $q = -k \dfrac{dT}{dx}$



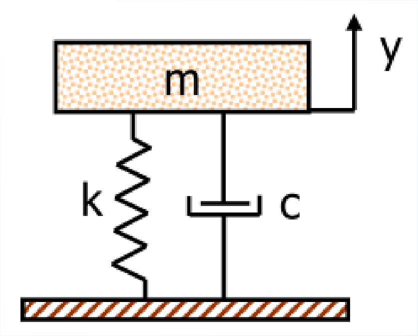
Swinging pendulum $\dfrac{d^2\theta}{dt^2} + \dfrac{g}{L}\sin(\theta) = 0$



Examples of ODE's

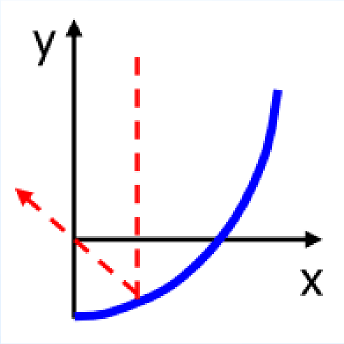
Mass-spring-damper system

$$m \frac{d^2y}{dt^2} + c \frac{dy}{dt} + k\,y = 0$$



Collector of a solar heater

$$x \left(\frac{dy}{dx}\right)^2 - 2\,y \frac{dy}{dx} = 0$$



TAYLOR SERIES

$$\frac{dy(t)}{dt} = f(t,y) \quad , \quad y(t_0) = y_0$$

$$y(t) = y(t_0) + (t - t_0)\, f(t_0,y_0) + \frac{(t - t_0)^2}{2!}\, y''(t_0) + \ldots + \frac{(t - t)^{n+1}}{(n+1)!}\, y^{(n+1)}(\xi)$$

$$y(t_1) = y(t_0) + (t_1 - t_0)\, f(t_0,y_0) + \frac{(t_1 - t_0)^2}{2!}\, y''(t_0) + \ldots + \frac{(t_1 - t_0)^{n+1}}{(n+1)!}\, y''(\xi)$$

$$y(t_2) = y(t_0) + (t_2 - t_0)\, f(t_0,y_0) + \frac{(t_2 - t_0)^2}{2!}\, y''(t_0) + \ldots + \frac{(t_2 - t_0)^{n+1}}{(n+1)!}\, y^{n+1}(\xi)$$

$$y(t_n) = y(t_0) + (t_n - t_0)\, f(t_0,y_0) + \frac{(t_n - t_0)^2}{2!}\, y''(t_0) + \ldots + \frac{(t_n - t_0)^{n+1}}{(n+1)!}\, y^{n+1}(\xi)$$

Too much error because t_0 and t_n are too far apart

RUNNING TAYLOR SERIES

$$\frac{dy(t)}{dt} = f(t,y) \quad , \quad y(t_0) = y_0$$

$$y(t) = y(t_0) + (t - t_0)\, f(t_0,y_0) + \frac{(t - t_0)^2}{2!}\, y''(t_0) + \ldots + \frac{(t - t)^{n+1}}{(n+1)!}\, y^{(n+1)}(\xi)$$

$$y(t_1) = y(t_0) + (t_1 - t_0)\, f(t_0,y_0) + \frac{(t_1 - t_0)^2}{2!}\, y''(t_0) + \ldots + \frac{(t_1 - t_0)^{n+1}}{(n+1)!}\, y^{n+1}(\xi)$$

$$y(t_2) = y(t_1) + (t_2 - t_1)\, f(t_1,y_1) + \frac{(t_2 - t_1)^2}{2!}\, y''(t_1) + \ldots + \frac{(t_2 - t_1)^{n+1}}{(n+1)!}\, y^{n+1}(\xi)$$

$$y(t_n) = y(t_{n-1}) + (t_n - t_{n-1})\, f(t_{n-1},y_{n-1}) + \frac{(t_n - t_{n-1})^2}{2!}\, y''(t_{n-1}) + \ldots + \frac{(t_n - t_{n-1})^{n+1}}{(n+1)!}\, y^{n+1}(\xi)$$

RUNNING TAYLOR SERIES

$$\frac{dy(t)}{dt} = f(t,y) \quad , \quad y(t_0) = y_0$$

$$y(t) = y(t_0) + (t - t_0)\, f(t_0,y_0) + \frac{(t - t_0)^2}{2!}\, y''(t_0) + \dots + \frac{(t - t)^{n+1}}{(n+1)!}\, y^{(n+1)}(\xi)$$

$$y(t_{n+1}) = y(t_n) + \underbrace{(t_{n+1} - t_n)}\, f(t_n,y_n) + \frac{(t_{n+1} - t_n)^2}{2!}\, y''(t_n) + \dots + \frac{(t_{n+1} - t_n)^{n+1}}{(n+1)!}\, y^{(n+1)}(\xi)$$

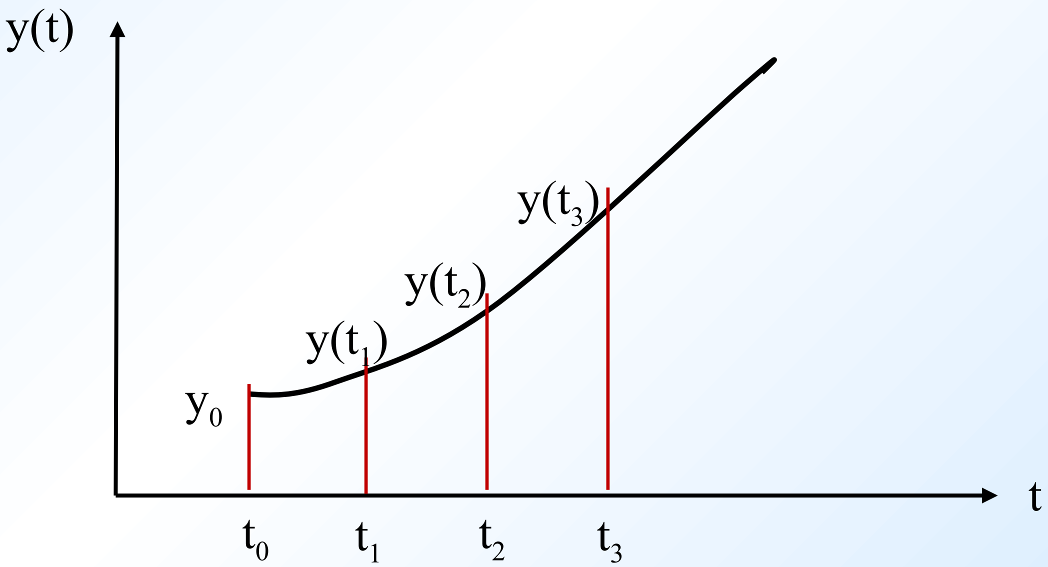
$$y_{n+1} = y_n + h\, f(t_n,y_n) + \frac{h^2}{2!}\, y''(t_n) + \dots + \frac{h^{n+1}}{(n+1)!}\, y^{(n+1)}(\xi)$$

$$y_{n+1} = y_n + h\, f(t_n,y_n) + \frac{h^2}{2!} \left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}\, f \right]_{t_n\,,\,y_n} + \dots + \frac{h^{n+1}}{(n+1)!}\, y^{(n+1)}(\xi)$$

Given: $\dfrac{dy(t)}{dt} = f\big(t, y(t)\big) \quad , \quad y(t_0) = y_0$

Question: $y(t) = ?$

Choose $h = \Delta t$ $\quad y(t_1) = ? \quad , \quad y(t_2) = ? \quad , \ \dots \ , \quad y(t_n) = ?$



$y(t_0) = y_0$ IC

$y(t_1) \cong y_1 = ?$

$y(t_2) \cong y_2 = ?$

...

$y(t_n) \cong y_n = ?$

Given: $$\frac{dy(t)}{dt} = f\big(t, y(t)\big) \quad , \quad y(t_0) = y_0$$

$$\int_{y_0}^{y} dy(t) = \int_{t_0}^{t} f\big(t, y(t)\big)\, dt$$

$$y(t) - y_0 = \int_{t_0}^{t} f\big(t, y(t)\big)\, dt \quad => \quad y(t) = y_0 + \underbrace{\int_{t_0}^{t} f\big(t, y(t)\big)\, dt}_{?}$$

$$y(t_{n+1}) - y(t_n) = \underbrace{\int_{t_n}^{t_{n+1}} f\big(t, y(t)\big)\, dt}_{?} \quad => \quad y_{n+1} = y_n + \int_{t_n}^{t_{n+1}} f\big(t, y(t)\big)\, dt$$

$$y_{n+1} = y_n + \int_{t_n}^{t_{n+1}} f\big(t, y(t)\big)\, dt \quad => \quad y_{n+1} = y_n + f\big(t_n, y_n\big) \underbrace{\int_{t_n}^{t_{n+1}} dt}_{\Delta t = h}$$

Leonhard Euler

Swiss Mathematician

1707 - 1783

Leonhard Euler (1707-1783) was one of the two greatest mathematicians of the post-Newton age, the other being Carl Friedrich Gauss. Euler was born in Basel, Switzerland, and educated at the University of Basel, at first with an eye toward following in his father's career as a Lutheran minister. With the assistance of his tutor and mentor Johann Bernoulli, however, he was able to convince his father to let him pursue a career in mathematics. In 1727 Euler joined the St. Petersburg Academy of Sciences in Russia, where he remained until 1741, at which time he joined the Berlin Academy of Sciences at the invitation of the Prussian king, Frederick the Great. After some disputes with the monarch, Euler left Berlin in 1766 and returned to St. Petersburg.

Euler's contributions to mathematics are almost unmatched in their breadth. He published an enormous amount of material, in a wide variety of areas, including infinite series, special functions (a field of study that he practically invented), number theory, complex variables, and hydrodynamics. His name is attached to countless results in mathematics, from Euler's formula relating the trigonometric functions to complex exponentials, to the Euler-Cauchy differential equations, to Euler's formula relating the number of sides, edges, and vertices in a polyhedron. His influence on notation is still felt today, as it was Euler who introduced e, π, and $i = \sqrt{-1}$ into the literature as standard symbols, in addition to the use of Σ for denoting summations, and cos and sin for the cosine and sine of an angle. Euler's collected works, published between 1911 and 1975, encompass 72 volumes!

The method for numerically solving differential equations that bears his name was apparently first presented in the period 1768-1769, in the two-volume work known as «Institutiones calculi integralis». The theoretical basis for the convergence of the method was laid down by Augustin Louis Cauchy in the mid-1800s and by Rudolf Lipschitz in the late 1800s.

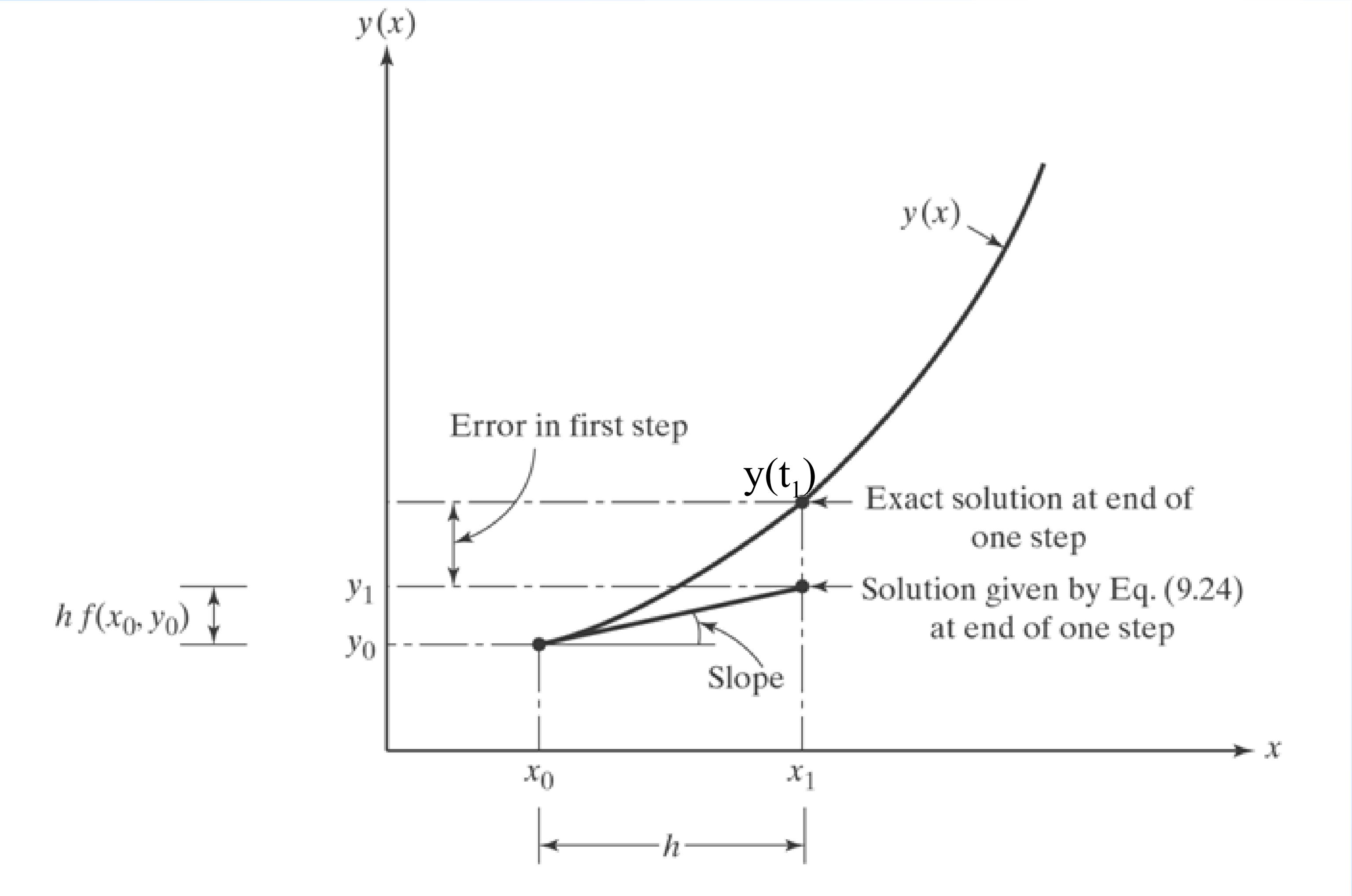
EULER'S METHOD: $y_{n+1} = y_n + h\, f(t_n, y_n)$

Local Error: $E_{local} = \frac{h^2}{2!} y''(\xi) \quad , \quad t_{n+1} < \xi < t_n$

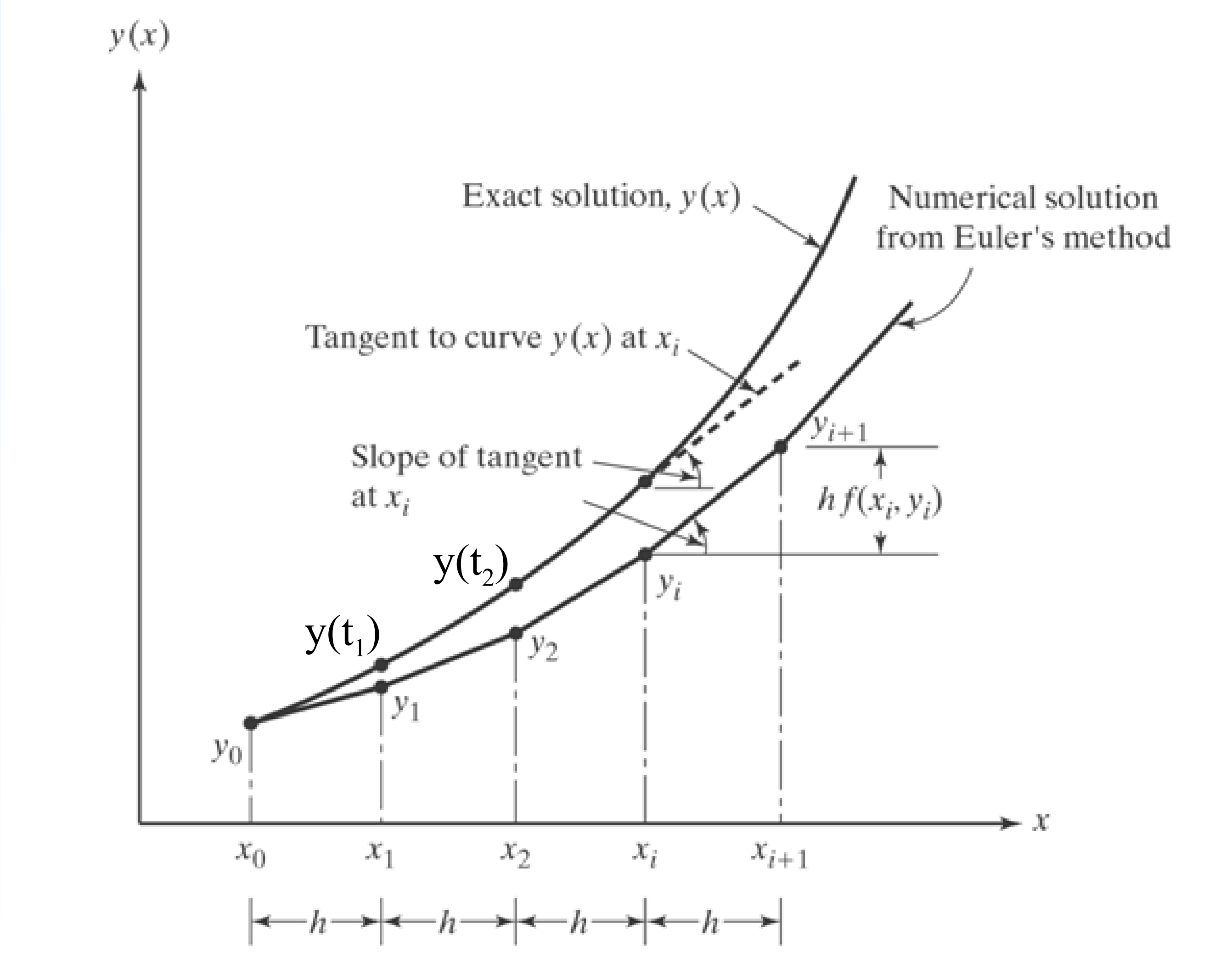
Total or Overall Error:

$$
\begin{aligned}
E_{total} &= \sum \frac{h^2}{2!} y''(\xi_n) \\
&= \sum \frac{h}{2!} y''(\xi_n)\, h \le \frac{h}{2} \text{Max}(y'') \sum h \\
&\le \frac{h}{2} \text{Max}(y'')\,(t_n - t_0) \\
&\propto h^1
\end{aligned}
$$

Euler's method is an order 1 method, O(1) or $O(h^1)$, i.e., the overall error is proportional to the first power of the step size, h.



Geometrical Interpretation of Euler's method



Accumulation of Error in Euler’s method

EXAMPLE ON EULER'S METHOD

$$\frac{d\,y\,(t)}{d\,t} = -\,a\,y\,(t) \quad , \quad y\,(0) = 1 = y_0$$

$$y_{exact} = e^{-a\,t}$$

Euler's Method: $y_{n+1} = y_n + h\,f\,(t_n\,,\,y_n)$

n	0	1	2	3	4	5	10	100	1000
t / a	0	0.1	0.2	0.3	0.4	0.5	1.0	10.0	100.0
y_{exact}	1	0.905	0.819	0.741	0.670	0.607	0.368	$4.5\ 10^{-5}$	$3.7\ 10^{-44}$
y_{Euler}	1	0.900	0.810	0.729	0.656	0.590	0.349	$2.6\ 10^{-5}$	$1.0\ 10^{-46}$

MODIFICATIONS OF EULER'S METHOD

Euler's Method: $y_{n+1} = y_n + h\,f\,(t_n\,,\,y_n)$

Mod. 1 $y_{n+1} = y_n + h\,f\,(t_{n+1}\,,\,y_{n+1})$ — f(t,y) is assumed to remain constant at the end of the interval

Mod. 2 $y_{n+1} = y_n + h\,f\left(t_n + \frac{h}{2}\,,\,\frac{y_n + y_{n+1}}{2}\right)$ — f(t,y) is assumed to remain constant at an average value

y_{n+1} is implicit, not explicit => root finding?

EXAMPLE ON MODIFICATIONS OF EULER'S METHOD

$$\frac{d\,y(t)}{d\,t} = -a\,y(t) \quad , \quad y(0) = 1$$

$$y_{exact} = e^{-a\,t}$$

n	0	1	2	3	4	5	10	100	1000
t / a	0	0.1	0.2	0.3	0.4	0.5	1.0	10.0	100.0
y_{exact}	1	0.905	0.819	0.741	0.670	0.607	0.368	$4.5\ 10^{-5}$	$3.7\ 10^{-44}$
y_{Euler}	1	0.900	0.810	0.729	0.656	0.590	0.349	$2.6\ 10^{-5}$	$1.0\ 10^{-46}$
y_{end}	1	0.909	0.826	0.751	0.683	0.621	0.386	$7.3\ 10^{-5}$	$4.1\ 10^{-42}$
y_{middle}	1	0.905	0.819	0.741	0.670	0.606	0.368	$4.6\ 10^{-5}$	$3.4\ 10^{-44}$

Runge-Kutta Methods

Running Taylor Series:

$$y_{n+1} = y_n + h\,y'(t_n) + \frac{h^2}{2!}\,y''(t_n) + \frac{h^3}{3!}\,y''(t_n) + \dots + \frac{h^{n+1}}{(n+1)!}\,y^{(n+1)}(\xi)$$

$$y_{n+1} = y_n + h\,f(t_n, y_n) + \frac{h^2}{2!}\left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}\,\frac{dy}{dt}\right]\Bigg|_{t_n\,,\,y_n} + \frac{h^3}{3!}\,y^{(3)}(\xi)$$

Runge-Kutta order two:

$$y_{n+1} = y_n + a\,h\,f(t_n, y_n) + b\,h\,f(t_n^*, y_n^*) + \frac{h^3}{3!}\,y^{(3)}(\xi)$$

$$t_n^* = t_n + \alpha\,h \qquad y_n^* = y_n + \beta\,h\,f(t_n, y_n)$$

Carl David Tolmé Runge

German Mathematician

1856 – 1927

Carl Runge (1856-1927) was bom in Bremen and educated at the University of Munich. He originally intended to study literature but after less than two months switched to mathematics and physics. In 1877 he began attending the University of Berlin, where he received a doctorate in 1880, on differential geometry. He held academic positions at Hanover and Göttingen and retired in 1925. Much of his professional work was more in physics than in mathematics, but he did make contributions in the numerical solution of differential equations (Runge-Kutta methods) and polynomial interpolation theory. The paper in which he outlined the theory behind the so-called "Runge example" appeared in 1901, under the title «Über empirische Funktionen und die Interpolation zwischen äquidistanten Ordinaten».

Martin Wilhelm Kutta (1867-1944) studied at Breslau and Munich, in addition to a year spent in Britain at Cambridge. Most of his professional career was spent in Stuttgart. Building on Runge's original idea (first presented in an 1894 article), Kutta published his version of the Runge-Kutta methods in 1901.

Runge-Kutta Methods

There are four unknowns: a, b, α, and β

Four relations are necessary to determine a, b, α, and β

Equate both sides, keeping the same local error:

$$y_n + h\, f_n + \frac{h^2}{2!}\left[\left.\frac{\partial f}{\partial t}\right|_{t_n, y_n} + \left.\frac{\partial f}{\partial y}\right|_{t_n, y_n} f_n \right] = y_n + a\, h\, f_n + b\, h\, f(t_n^*, y_n^*)$$

Expand $f(t_n^*, y_n^*)$ in terms of Taylor series around (t_n, y_n) and

equate the coefficients of the same powers of h. This gives three relations, not four

$a + b = 1$

$\alpha\, b = 1/2$

$\beta\, b = 1/2$

Choose one of the fractions arbitrarirly, that sets the other three unknowns.

Second-Order Runge-Kutta Methods

Heun's Method:

$$y_{n+1} = y_n + \frac{h}{2}\left[f(t_n, y_n) + f\left(t_{n+1}, y_n + h f(t_n, y_n)\right)\right]$$

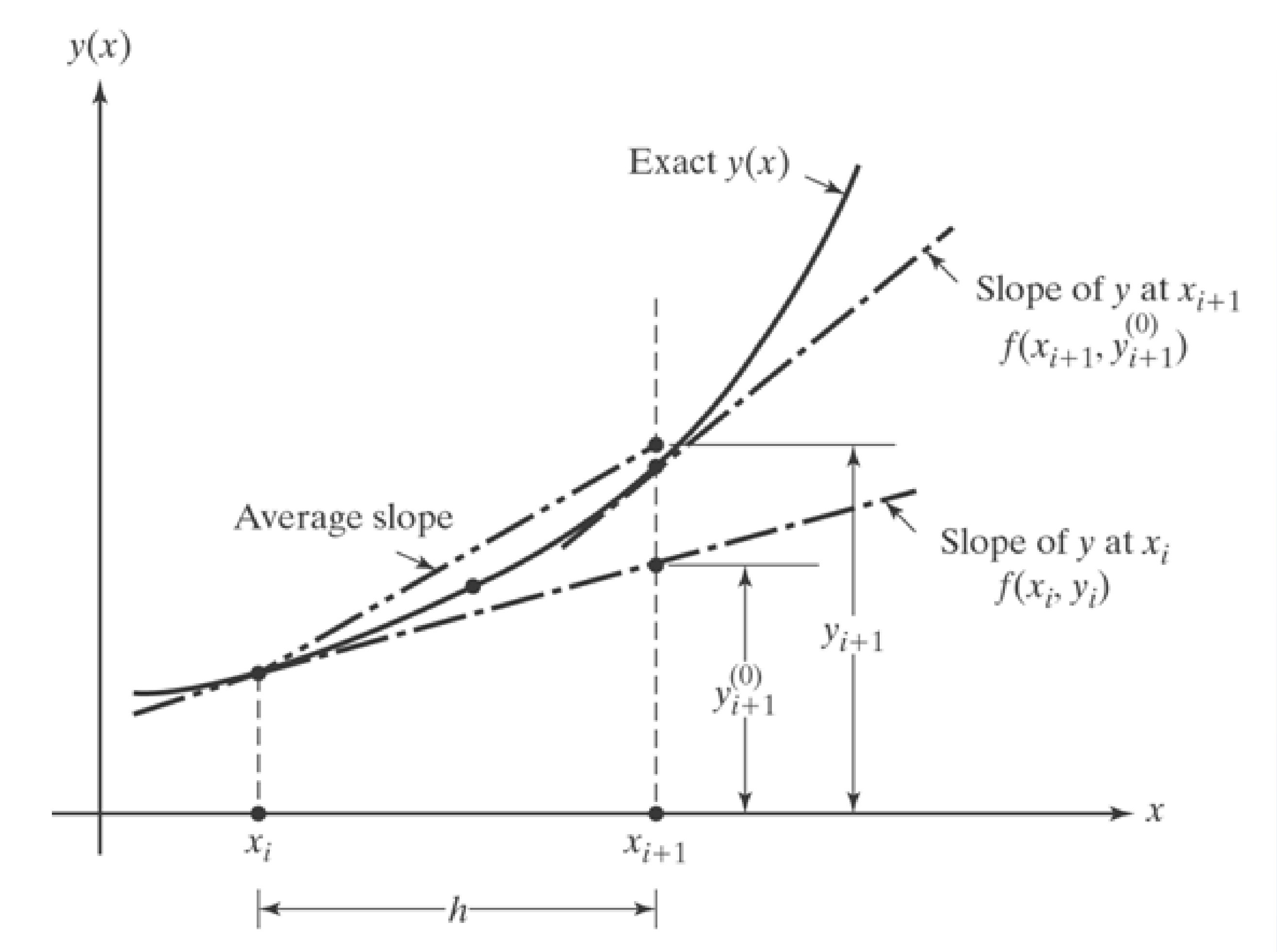
The Modified Euler's Method:

$$y_{n+1} = y_n + h\left[f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2} f(t_n, y_n)\right)\right]$$

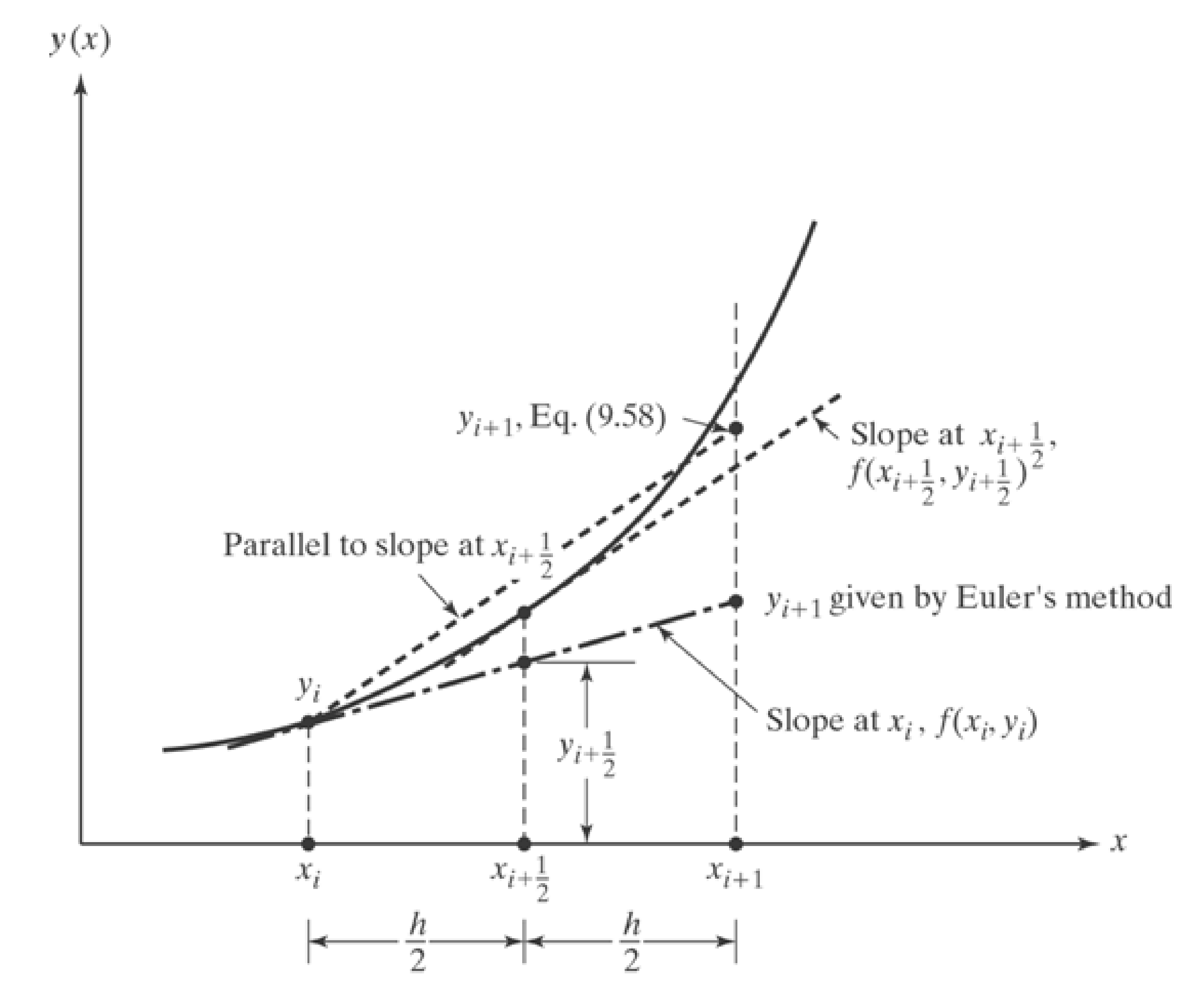
Karl Heun

German Mathematician

1859 - 1929



Geometrical Interpretation of Heun's method



Geometrical Interpretation of the Modified Euler's method

Running Taylor Series:

$$y_{n+1} = y_n + h\,y'(t_n) + \frac{h^2}{2!}\,y''(t_n) + \frac{h^3}{3!}\,y'''(t_n) + \frac{h^4}{4!}\,y^{(4)}(t_n) + \underbrace{\frac{h^5}{5!}\,y^{(5)}(\xi)}_{\text{Local Error}}$$

Runge-Kutta order 4:

$$y_{n+1} = y_n + a\,h\,f(t_n,y_n) + b\,h\,f(t_n^{*},y_n^{*}) + c\,h\,f(t_n^{**},y_n^{**}) + c\,h\,f(t_n^{***},y_n^{***}) + \overbrace{\frac{h^5}{5!}\,y^{(5)}(\xi)}$$

$$t_n^{*} = t_n + \alpha\,h \qquad y_n^{*} = y_n + \beta\,h\,f(t_n,y_n)$$

$$t_n^{**} = t_n + \gamma\,h \qquad y_n^{**} = y_n + \eta\,h\,f(t_n,y_n)$$

$$t_n^{***} = t_n + \nu\,h \qquad y_n^{**} = y_n + \varepsilon\,h\,f(t_n,y_n)$$

- 10 unknowns
- Equate the Coefficients of the same power of h
- 9 relations

Fourth-Order Runge-Kutta Methods

Method Attributed to Runge:

$$y_{n+1} = y_n + \frac{h}{6}\left(K_1 + 2K_2 + 2K_3 + K_4\right)$$

$$K_1 = f\,(t_n\,, y_n)$$

$$K_2 = f\left(t_n + \frac{h}{2}\,,\; y_n + \frac{h}{2}K_1\right)$$

$$K_3 = f\left(t_n + \frac{h}{2}\,,\; y_n + \frac{h}{2}K_2\right)$$

$$K_4 = f\,(t_n + h\,,\; y_n + h\,K_3)$$

This method reduces to Simpson's one-third rule when $f\,(t,y) = f\,(t)$.

Method Attributed to Kutta:

$$y_{n+1} = y_n + \frac{h}{8}\left(K_1 + 3K_2 + 3K_3 + K_4\right)$$

$$K_1 = f\,(t_n\,, y_n)$$

$$K_2 = f\left(t_n + \frac{h}{3}\,,\; y_n + \frac{h}{3}K_1\right)$$

$$K_3 = f\left(t_n + \frac{2h}{3}\,,\; y_n - \frac{h}{3}K_1 + h\,K_2\right)$$

$$K_4 = f\,(t_n + h\,,\; y_n + h\,K_1 - h\,K_2 + h\,K_3)$$

This method reduces to Simpson's 3/8 rule when $f(t,y) = f(t)$.

Runge-Kutta-Gill Method:

$$y_{n+1} = y_n + \frac{h}{6}\left(K_1 + 2\left(1 - \frac{1}{\sqrt{2}}\right)K_2 + 2\left(1 + \frac{1}{\sqrt{2}}\right)K_3 + K_4\right)$$

$$K_1 = f\,(t_n\,,y_n)$$

$$K_2 = f\left(t_n + \frac{h}{2}\;,\; y_n + \frac{h}{2}K_1\right)$$

$$K_3 = f\left(t_n + \frac{h}{2}\;,\; y_n + \left(-\frac{1}{2} + \frac{1}{\sqrt{2}}\right)h\,K_1 + \left(1 - \frac{1}{\sqrt{2}}\right)h\,K_2\right)$$

$$K_4 = f\,(t_n + h\;,\; y_n - \frac{1}{\sqrt{2}}h\,K_2 + \left(1 + \frac{1}{\sqrt{2}}\right)h\,K_3)$$

This is one of the most widely used fourth-order methods. The constants are selected to reduce the amount of storage required in the solution of a large number of simultaneous first-order differential equations.

EXAMPLE ON RUNGE-KUTTA METHODS

Euler's Method: $y_{n+1} = y_n + h\, f(t_n, y_n)$

Heun's Method $y_{n+1} = y_n + \frac{h}{2}\left[f(t_n, y_n) + f\left(t_{n+1}, y_n + h\, f(t_n, y_n)\right)\right]$

The Runge-Kutta Order 4 $y_{n+1} = y_n + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$K_1 = f(t_n, y_n)$

$K_2 = f(t_n + \frac{h}{2}, y_n + \frac{h}{2} K_1)$

$K_3 = f(t_n + \frac{h}{2}, y_n + \frac{h}{2} K_2)$

$K_4 = f(t_{n+1}, y_n + h K_3)$

$$\frac{d\,y(t)}{d\,t} = -a\,y(t) \quad , \quad y(0) = 1$$

$$y_{exact} = e^{-a\,t}$$

n	0	1	2	3	4	5	10	100	1000
t / a	0	0.1	0.2	0.3	0.4	0.5	1.0	10.0	100.0
$\mathbf{y}_{\mathbf{exact}}$	1	0.90484	0.81873	0.74082	0.67032	0.60653	0.36788	$4.54\ 10^{-5}$	$3.72\ 10^{-44}$
$\mathbf{y}_{\mathbf{Euler}}$	1	0.900	0.810	0.729	0.656	0.590	0.349	$2.6\ 10^{-5}$	$1.0\ 10^{-46}$
$\mathbf{y}_{\mathbf{R\text{-}K\ 2}}$	1	0.905	0.819	0.741	0.671	0.607	0.369	$4.6\ 10^{-5}$	$4.45\ 10^{-44}$
$\mathbf{y}_{\mathbf{R\text{-}K\ 4}}$	1	0.90484	0.81873	0.74082	0.67032	0.60653	0.36788	$4.54\ 10^{-5}$	$3.72\ 10^{-44}$

STABILITY OF RUNGE-KUTTA METHODS

$$\frac{dy}{dt} = \lambda y \qquad \text{IC: } y(0) = 1$$

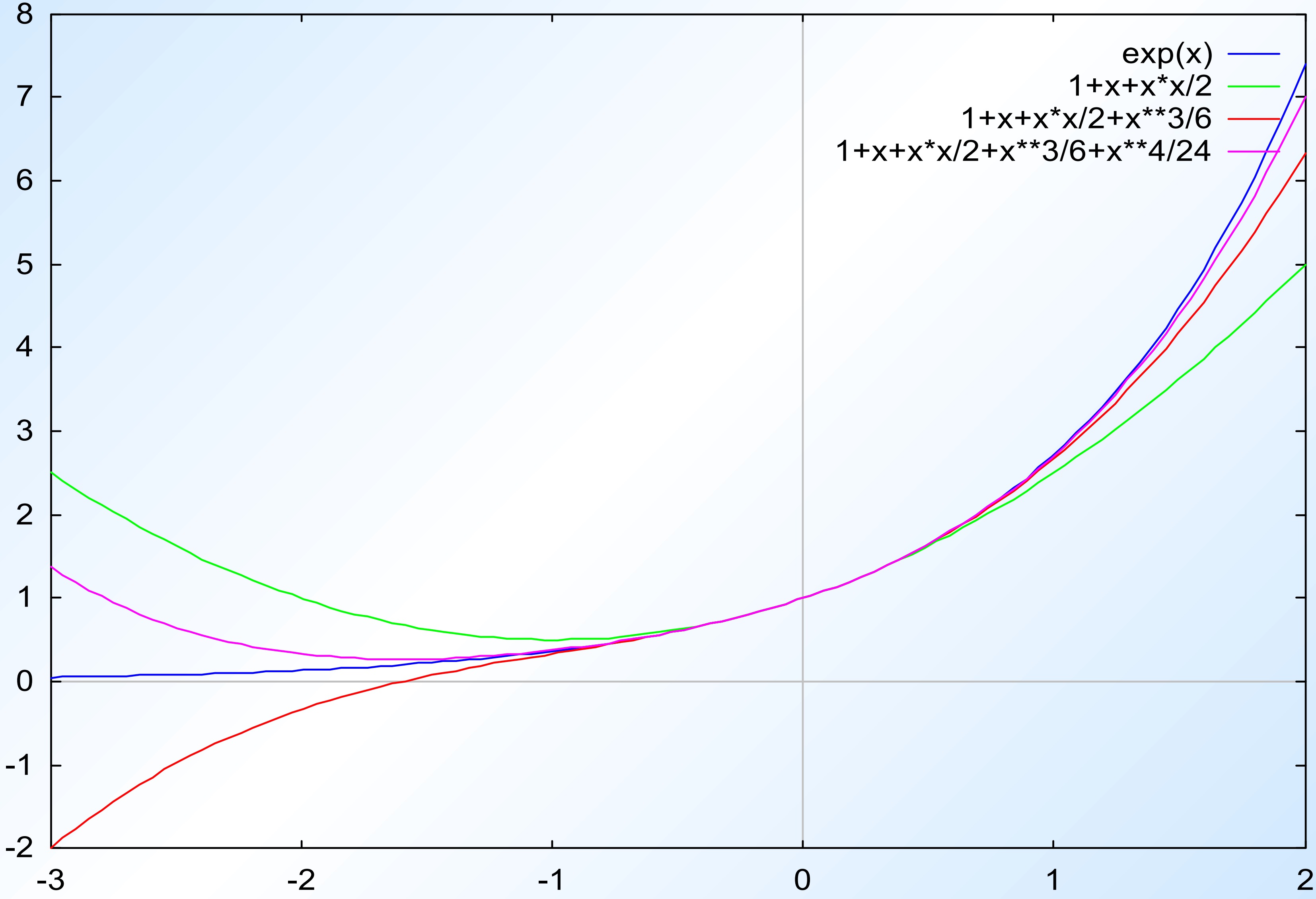
Exact Solution: $y(t) = e^{\lambda t}$

Numerical Solution: $y_{n+1} = y_n P(\lambda h)$

Euler's Method $$y_{n+1} = y_n (1 + \lambda h)$$

Runge-Kutta Order 2 $$y_{n+1} = y_n \left(1 + \lambda h + \frac{(\lambda h)^2}{2}\right)$$

Runge-Kutta Order 4 $$y_{n+1} = y_n \left(1 + \lambda h + \frac{(\lambda h)^2}{2} + \frac{(\lambda h)^3}{3!} + \frac{(\lambda h)^4}{4!}\right)$$


exp(x)
1+x+x*x/2
1+x+x*x/2+x**3/6
1+x+x*x/2+x**3/6+x**4/24
8
7
6
5
4
3
2
1
0
-1
-2
-3
-2
-1
0
1
2

EXAMPLE ON STABILITY OF RUNGE-KUTTA METHODS

Given: $$\frac{dy}{dt} = 10y - 10t - 1$$

General Solution $$y(t) = A e^{10t} - t$$

Initial Condition $$y(0) = 0$$

Exact Solution: $$y(t) = -t \quad , \quad A = 0$$

t	Exact	Runge-Kutta Order 4		
		h = 0.01		
0.2	- 0.2	- 0.2		
0.4	- 0.4	- 0.39998		
0.6	- 0.6	- 0.59986		
0.8	- 0.8	- 0.79896		
1.0	- 1.0	- 0.99232		
1.2	- 1.2	- 0.11134		
1.4	- 1.4	- 0.98318		
1.6	- 1.6	1.4702		
1.8	- 1.8	20.815		
2.0	- 2.0	164.59		

t	Exact	Runge-Kutta Order 4		
		h = 0.01	h = 0.001	
0.2	- 0.2	- 0.2	- 0.19998	
0.4	- 0.4	- 0.39998	- 0.39988	
0.6	- 0.6	- 0.59986	- 0.59918	
0.8	- 0.8	- 0.79896	- 0.79394	
1.0	- 1.0	- 0.99232	- 0.95529	
1.2	- 1.2	- 0.11134	- 0.86994	
1.4	- 1.4	- 0.98318	1.0378	
1.6	- 1.6	1.4702	16.410	
1.8	- 1.8	20.815	131.26	
2.0	- 2.0	164.59	981.07	

t	Exact	Runge-Kutta Order 4		
		h = 0.01	h = 0.001	h = 0.0001
0.2	- 0.2	- 0.2	- 0.19998	- 0.19986
0.4	- 0.4	- 0.39998	- 0.39988	- 0.39920
0.6	- 0.6	- 0.59986	- 0.59918	- 0.59483
0.8	- 0.8	- 0.79896	- 0.79394	- 0.76311
1.0	- 1.0	- 0.99232	- 0.95529	- 0.72925
1.2	- 1.2	- 0.11134	- 0.86994	0.79465
1.4	- 1.4	- 0.98318	1.0378	13.316
1.6	- 1.6	1.4702	16.410	107.03
1.8	- 1.8	20.815	131.26	800.08
2.0	- 2.0	164.59	981.07	5921.3

A GENERAL DICTUM IN NUMERICAL MATH

If anything at all is known about the errors in a process, that knowledge can be exploited to improve the process.

$$\phi(h) = L - \sum_{k=1}^{\infty} a_{2k}\, h^{2k}$$

L: Objective to be evaluated

ϕ (h): Known function of chosen h

$$D(n,1) = \phi\left(\frac{h}{2^n}\right) \qquad n = 1, 2, 3, \ldots$$

$$D(n,1) = L + \sum_{k=1}^{\infty} A(k,1) \left(\frac{h}{2^n}\right)^{2k}$$

Richardson's Extrapolation

$$D(n, m+1) = \frac{4^m}{4^m - 1} D(n, m) - \frac{1}{4^m - 1} D(n-1, m)$$

D (1 , 1)			
D (2 , 1)	D (2 , 2)		
D (3 , 1)	D (3 , 2)	D (3 , 3)	
D (4 , 1)	D (4 , 2)	D (4 , 3)	D (4 , 4)

- Accuracy increases as we move down the table in any column because h is decreased by half at each step.
- Accuracy increases as we move to the right in any row because of order-of-magnitude decrease in h.
- Note that smaller h does not always leads to higher accyracy. The accumulation of round-off error eventually catches up with the process.

Example on Richardson's Extrapolation

Solving Integrals

Exact solution: $$\int_0^4 f(x)\ dx = \int_0^4 2^x\ dx = 21.640426$$

Richardson's extrapolation table:

One Panel (n = 1)	34.0		
Two Panels (n = 2)	25.0	22.0	
Four Panels (n = 4)	22.5	21.666	21.6444

Percent Errors:

One Panel (n = 1)	- 57.1 %		
Two Panels (n = 2)	- 15.5 %	- 1.7 %	
Four Panels (n = 4)	- 4.0 %	- 0.12 %	- 0.02 %

Example on Richardson's Extrapolation

Solving ODE

Given: $$\frac{dy}{dt} = -y \quad , \quad y(0) = 1$$

Use Euler's method: $$h = \frac{0.25}{2^{k-1}} \quad , \quad k = 1, 2, 3, \ldots$$

Exact solution: y (0.25) = 0.7788008

n	Richardson's extrapolation table			
1	0.75			
2	0.765625	0.78125		
4	0.772476	0.7793274	0.7786865	
8	0.7756999	0.7789236	0.778789	0.7788036

n	Percent Errors			
1	3.7			
2	1.7	- 0.3		
4	0.8	- 0.07	0.015	
8	0.4	- 0.02	0.0015	- 0.00036

Selecting step size, h:

Given ODE: $$\frac{dy}{dt} = f(t, y) \quad , \quad y(t_0) = y_0$$

Exact (unknown) solution at t_n: $$y(t_n) = y^*$$

Local error from step t_{n-1} to t_n with step size h_1: $$E = C\, h_1^{p+1}$$

Numerical solution of order p at t_n: $$y_{n, h_1} = y^* + C\, h_1^{p+1}$$

Repeat the numerical solution with step size : $$h_2 = \frac{h_1}{2}$$

Numerical solution of order p at t_n: $$y_{n, h_2} \cong y^* + 2\, C\, h_2^{p+1}$$

$$y_{n,h_1} = y^* + C\,h_1^{p+1} \qquad y_{n,h_2} \cong y^* + 2\,C\,h_2^{p+1}$$

Combine these two equations and solve for y^*:

$$y^* = \frac{y_{n,h_1} - 2^p\, y_{n,h_2}}{1 - 2^p}$$

Error in $y_{n,h1}$:

$$E = y_n^* - y_{n,h_1} = \frac{2^p \left(y_{n,h_2} - y_{n,h_1}\right)}{2^p - 1} = C\,h_1^{p+1}$$

$$C = \frac{2^p}{2^p - 1}\,\frac{y_{n,h_2} - y_{n,h_1}}{h_1^{p+1}}$$

$$E_{max} = C\,h_{max}^{p+1} = \frac{2^p}{2^p - 1}\,\frac{y_{n,h_2} - y_{n,h_1}}{h_1^{p+1}}\,h_{max}^{p+1} \le \varepsilon$$

$$h_{max} \le h_1 \left[\frac{\varepsilon\,\left(2^p - 1\right)}{2^p\,\left|y_{n,h_2} - y_{n,h_1}\right|}\right]^{\frac{1}{p+1}}$$

Maximum step size for local error not to exceed ε.

The above procedure is rather expensive when we want to check h at every step of the calculation. We need to have an inexpensive method with less number of calculations.

Consider two methods of order p and q such that $p+1 \le q$

Define:

$$y_{n,p} = y_n^* + A_{p+1}\, h^{p+1} + O(h^{p+2})$$

$$y_{n,q} = y_n^* + B_{q+1}\, h^{q+1} + O(h^{q+2})$$

Subtract side by side: $$y_{n,q} - y_{n,p} = A_{p+1}\, h^{p+1} + O(h^{p+2})$$

Therefore: $$E_n = y_{n,q} - y_{n,p}$$ This is the local error in $y_{n,p}$

Runge-Kutta-Fehlberg Order 4

$$y_{n+1} = y_n + h\left(\frac{25}{216}F_1 + \frac{1408}{2565}F_3 + \frac{2197}{4104}F_4 - \frac{1}{5}F_5\right)$$

$$F_1 = f\,(t_n\,,y_n)$$

$$F_2 = f\left(t_n + \frac{1}{4}h\,,\,y_n + \frac{1}{4}h\,F_1\right)$$

$$F_3 = f\left(t_n + \frac{3}{8}h\;,\;y_n + \frac{3}{32}h\,F_1 + \frac{9}{32}h\,F_2\right)$$

$$F_4 = f\left(t_n + \frac{12}{13}h\;,\;y_n + \frac{1932}{2197}h\,F_1 - \frac{7200}{2197}h\,F_2 + \frac{7296}{2197}h\,F_3\right)$$

$$F_5 = f\left(t_n + h\;,\;y_n + \frac{439}{216}h\,F_1 - 8\,h\,F_2 + \frac{3680}{513}h\,F_3 - \frac{845}{4104}h\,F_4\right)$$

Runge-Kutta-Fehlberg Order 5

$$y_{n+1} = y_n + h\left(\frac{16}{135}F_1 + \frac{6656}{12825}F_3 + \frac{28561}{56430}F_4 - \frac{9}{50}F_5 + \frac{2}{55}F_6\right)$$

$$F_1 = f\,(t_n\,,y_n)$$

$$F_2 = f\left(t_n + \frac{1}{4}h\,,\,y_n + \frac{1}{4}h\,F_1\right)$$

$$F_3 = f\left(t_n + \frac{3}{8}h\ ,\ y_n + \frac{3}{32}h\,F_1 + \frac{9}{32}h\,F_2\right)$$

$$F_4 = f\left(t_n + \frac{12}{13}h\ ,\ y_n + \frac{1932}{2197}h\,F_1 - \frac{7200}{2197}h\,F_2 + \frac{7296}{2197}h\,F_3\right)$$

$$F_5 = f\left(t_n + h\ ,\ y_n + \frac{439}{216}h\,F_1 - 8\,h\,F_2 + \frac{3680}{513}h\,F_3 - \frac{845}{4104}h\,F_4\right)$$

$$F_6 = f\left(t_n + \frac{1}{2}h\ ,\ y_n - \frac{8}{27}h\,F_1 + 2\,h\,F_2 - \frac{3544}{2565}h\,F_3 + \frac{1859}{4104}h\,F_4 - \frac{11}{40}h\,F_5\right)$$

Multi-step Methods

Multi-step (multi-procedure) methods require information on the solution of more than one point, and hence they are **not self-starting**.

- They are however more efficient (faster) than R-K methods.

- You may even hope that they may solve the instability problem of R-K methods.

Example using Euler's method:

$$\frac{dy(t)}{dt} = f\big(t, y(t)\big) \quad , \quad y(t_0) = y_0 \qquad \Rightarrow \qquad y_{n+1} = y_n + h\, f(t_n, y_n) + \underbrace{O(h^2)}_{\text{Local error}}$$

$$y_1 \cong y_0 + h\, f(t_0, y_0)$$

$$y_2 \cong y_1 + h\, f(t_1, y_1)$$

etc.

If y_1 is also known (besides y_0), what can you do for a better estimation of y_2?

Example using Euler's method:

$$\frac{dy(t)}{dt} = f\left(t,y(t)\right) \quad , \quad y(t_0) = y_0 \quad \text{and} \quad \underbrace{y(t_1) \cong y_1}$$

$$y_2 = ?$$

Somehow known very accurately, maybe from a very accurate numerical calculation

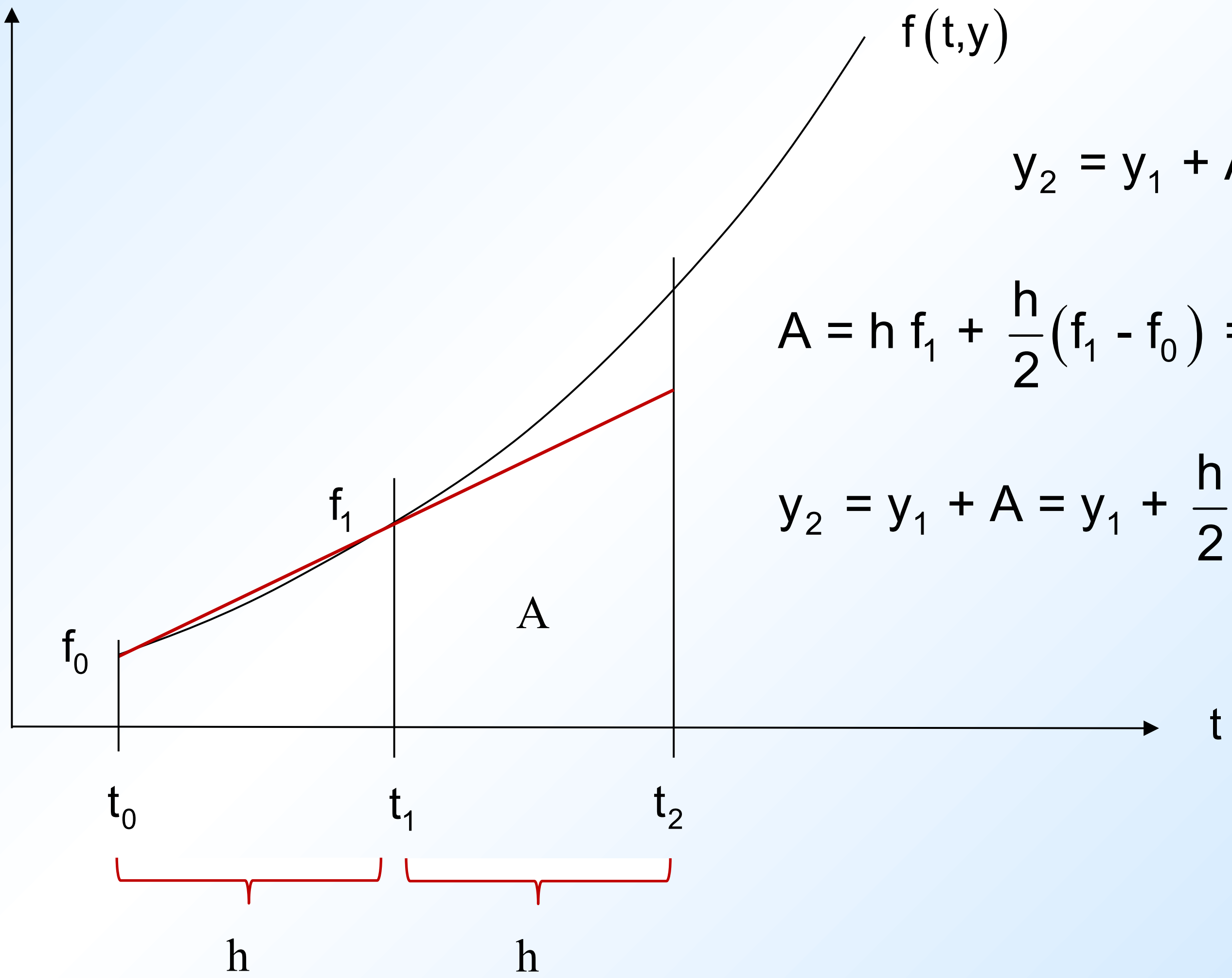
Find y_2 (and the rest of the points, y_3, y_4, etc.) fitting a first degree polynomial to $f(t_0,y_0)$ and $f(t_1,y_1)$, extend to the next panel, and find the area.

$$y_2 = y_1 + \frac{h}{2}\left(3\, f(t_1,y_1) - f(t_0,y_0)\right) = y_1 + \frac{h}{2}\,(3\, f_1 - f_0)$$

$$y_{n+1} = y_n + \frac{h}{2}\,(3\, f_n - f_{n-1}) \quad , \quad n = 1, 2, 3, ...$$

Local truncation error: $E_{local} = \frac{5}{12}\, h^3\, f''(\xi)$ How do you find this?

$$\frac{dy}{dt} = f\left(t,y\right)$$



$$y_2 = y_1 + A$$

$$A = h\, f_1 + \frac{h}{2}\left(f_1 - f_0\right) = \frac{h}{2}\left(3\, f_1 - f_0\right)$$

$$y_2 = y_1 + A = y_1 + \frac{h}{2}\left(3\, f_1 - f_0\right)$$

Adams-Bashfort Open Formulae:

Order two: $y_{n+1} = y_n + \frac{h}{2}(3 f_n - f_{n-1})$, $n = 1, 2, 3, ...$

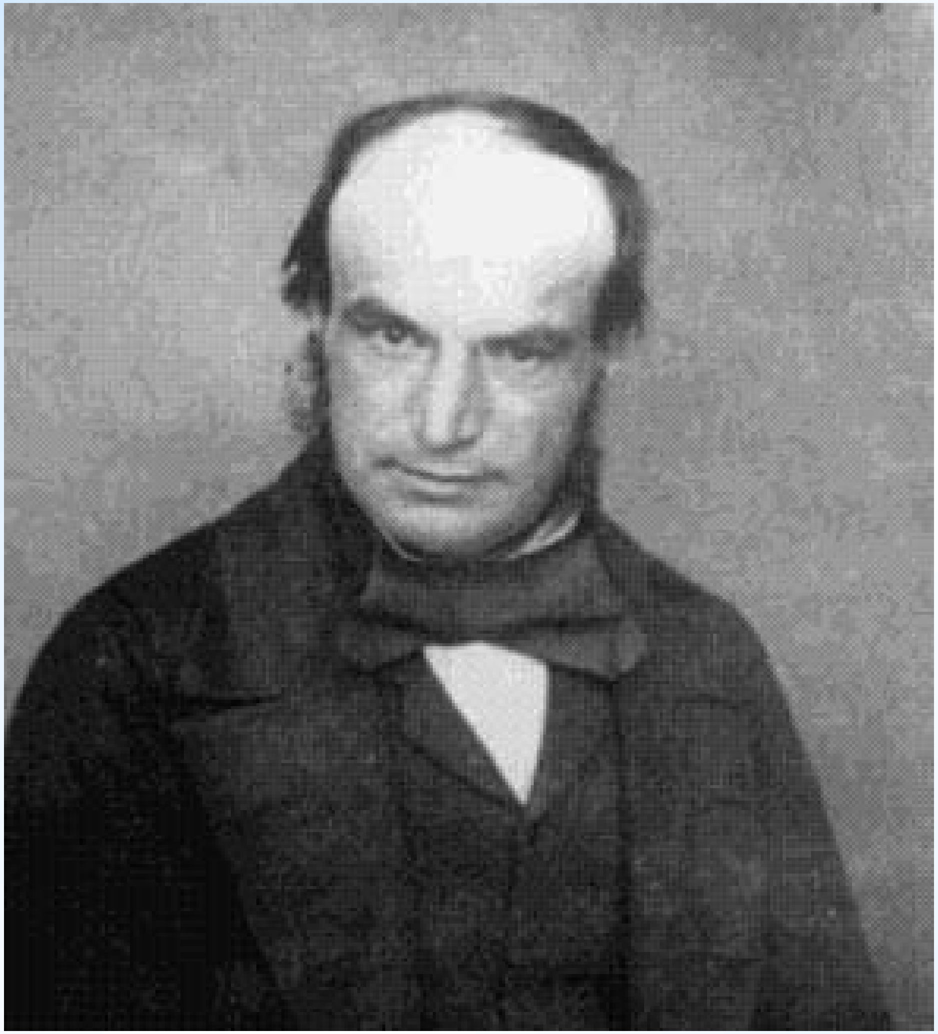
Local truncation error: $\frac{5}{12} h^3 f''(\xi)$ => $O(2)$

Order three: $y_{n+1} = y_n + \frac{h}{12}(23 f_n - 16 f_{n-1} + 5 f_{n-2})$, $n = 2, 3, 4, ...$

Local truncation error: $\frac{9}{24} h^4 f'''(\xi)$ => $O(3)$

Order four: $y_{n+1} = y_n + \frac{h}{24}(55 f_n - 59 f_{n-1} + 37 f_{n-2} - 9 f_{n-3})$, $n = 3, 4, 5, ...$

Local truncation error: $\frac{251}{720} h^5 f^{(4)}(\xi)$ => $O(4)$



John Couch Adams

British Astronomer

1819 - 1892

John Couch Adams (1819-1892) was born in Cornwall, England, and educated at St. John's College, Cambridge. Something of a child prodigy in mathematics, at Cambridge he compiled an extraordinary record and was awarded several prizes. While still an undergraduate he decided to study the irregularities in the orbit of the planet Uranus, to see if they could be explained by the gravitational attraction of an as-yet unknown eighth planet. Adams predicted the new planet's position, but, probably because of his youth, the Cambridge Observatory took no action so the credit for the discovery of Neptune went to Urbain Le Verrier, although the question of priority here is still controversial. Adams briefly held a position as professor of mathematics at St. Andrews College before being named Professor of Astronomy and director of the Cambridge Observatory.

Francis Bashforth

British Mathematician

1819 - 1912

Francis Bashforth (1819-1912) was born in Thumscoe, England, the son of a farmer, and attended St. John's College of Cambridge at the same time as Adams. Although he had been ordained as an Anglican priest in 1851, upon graduation Bashforth worked first as a civil engineer and surveyor for a railroad company, and then (in 1864) obtained a position as professor of applied mathematics at what evolved into the Royal Artillery College. Although Bashforth made numerous important contributions to the study of ballistics, in 1872 an army reorganization left him with such a reduced position that he resigned and became a parish rector. The Adams-Bashforth method comes from a joint study of capillary action that the two men wrote in 1883.

ADAMS-BASHFORT OPEN FORMULAE

$$y_{i+1} = y_i + h \sum_{k=1}^{n} \alpha_{nk} f_{i-k+1} + O(h^{n+1})$$

Order of the Formula (n)	k = 1 α_{n1}	k = 2 α_{n2}	k = 3 α_{n3}	k = 4 α_{n4}	k = 5 α_{n5}	k = 6 α_{n6}	Local Truncation Error, $O(h^{n+1})$
1	1						$\frac{1}{2} h^2 f'(\xi)$
2	$\frac{3}{2}$	$-\frac{1}{2}$					$\frac{5}{12} h^3 f''(\xi)$
3	$\frac{23}{12}$	$-\frac{16}{12}$	$\frac{5}{12}$				$\frac{9}{24} h^4 f'''(\xi)$
4	$\frac{55}{24}$	$-\frac{59}{24}$	$\frac{37}{24}$	$-\frac{9}{24}$			$\frac{251}{720} h^5 f^{(4)}(\xi)$
5	$\frac{1901}{720}$	$-\frac{2774}{720}$	$\frac{2616}{720}$	$-\frac{1274}{720}$	$\frac{251}{720}$		$\frac{475}{1440} h^6 f^{(5)}(\xi)$
6	$\frac{4277}{1440}$	$-\frac{7923}{1440}$	$\frac{9982}{1440}$	$-\frac{7298}{1440}$	$\frac{2877}{1440}$	$-\frac{475}{1440}$	$\frac{19087}{60480} h^7 f^{(6)}(\xi)$

EXAMPLE ON MULTI-STEP METHODS

$\dfrac{dy}{dt} = y\cos(t)$	IC: $y(0) = 1$
Exact Solution:	$y(t) = e \sin(t)$

Use Heun's method to start and 3-point Adams-Bashfort open formula to proceed:

Heun's Method

$$y_{n+1} = y_n + \frac{h}{2}\left[f(t_n , y_n) + f\left(t_{n+1} , y_n + h\, f(t_n , y_n)\right)\right]$$

Adams-Bashfort open order 3:

$$y^p_{n+1} = y_n + \frac{h}{12}\left(23\, f_n - 16\, f_{n-1} + 5\, f_{n-2}\right)$$

t	y (t)	Heun's			
	Exact	y_n	% Error		
0	0	1.0			
0.25	1.28070	1.27639	0.34		
0.5	1.61515	1.60492	0.63		
0.75	1.97712	1.95996	0.87		
1.0	2.31978	2.29591	1.03		
1.25	2.58309	2.55358	1.14		
1.5	2.71148	2.67858	1.21		
1.75	2.67510	2.64153	1.25		
2.0	2.48258	2.45139	1.26		

t	y (t)	Heun's		Adams-Bashfort	
	Exact	y_n	% Error	y_n	% Error
0	0	1.0		1.0	
0.25	1.28070	1.27639	0.34		
0.5	1.61515	1.60492	0.63		
0.75	1.97712	1.95996	0.87	1.97172	0.27
1.0	2.31978	2.29591	1.03	2.32236	- 0.11
1.25	2.58309	2.55358	1.14	2.58942	- 0.25
1.5	2.71148	2.67858	1.21	2.71148	- 0.04
1.75	2.67510	2.64153	1.25	2.66317	0.45
2.0	2.48258	2.45139	1.26	2.45679	1.04

Note that, the order of the starter method is better be greater than the order of the rest of the solution in order to reduce the beginning errors which may accumulate later on.

This is not so in the above example.

Example: Error calculation of Adams-Bashfort open formulae:

Prove the following

$$y_{n+1} = y_n + \frac{h}{2}\left[3 f_n - f_{n-1}\right] + \frac{5}{12} h^3 f''(\xi)$$

$$y_{n+1} = y_n + \frac{h}{12}\left[23 f_n - 16 f_{n-1} + 5 f_{n-2}\right] + \frac{9}{24} h^4 f'''(\xi)$$

Predictor-Corrector Procedures

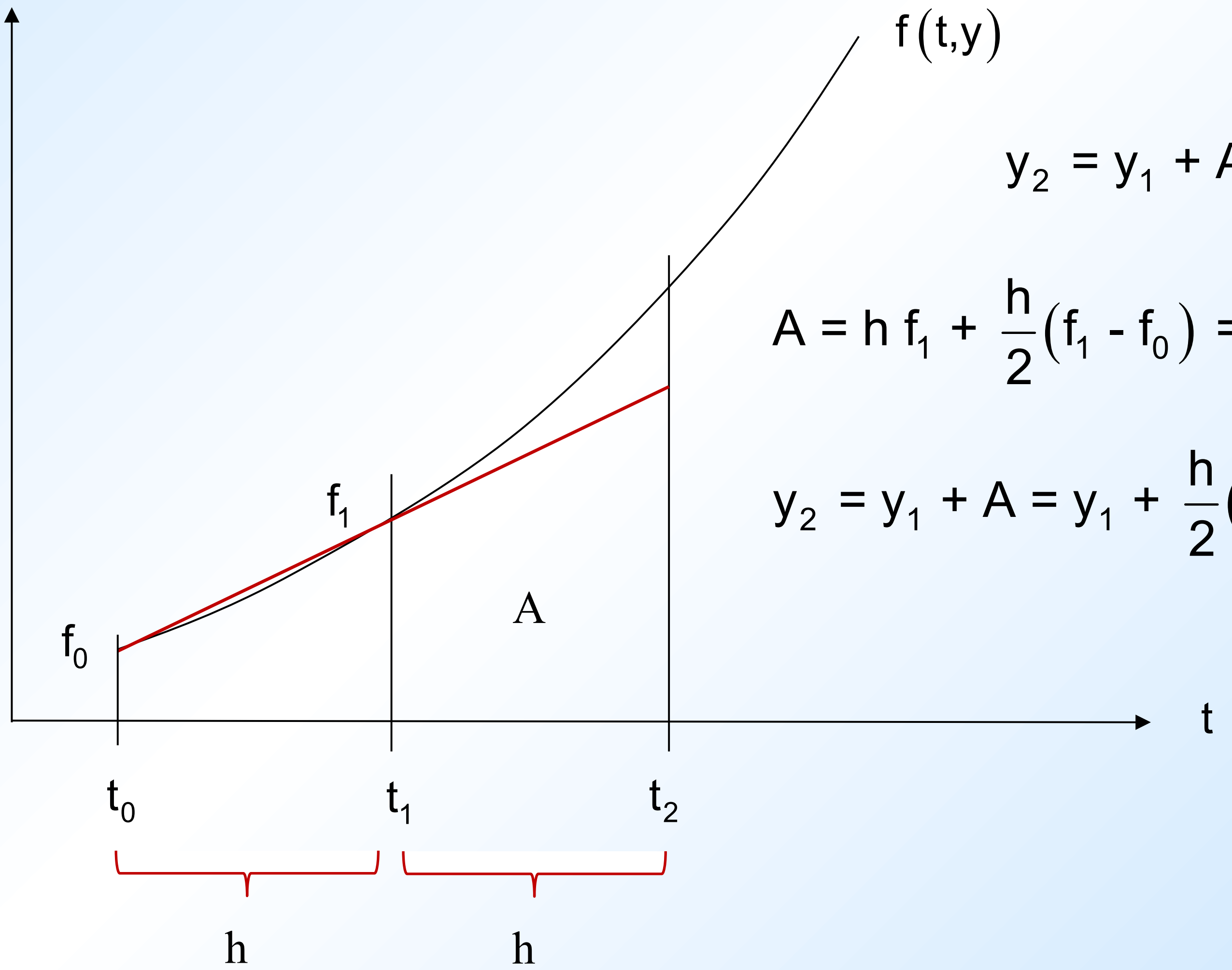
The idea behind the predictor-corrector methods is to use a suitable combination of an explicit and an implicit technique to obtain a method with better convergence characteristics.

Here is an example using Euler's method:

Predictor: $$y_{n+1}^{p} = y_n + h\,f(t_n, y_n) = y_n + h\,f_n$$

Corrector: $$y_{n+1}^{c} = y_n + \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1}^{p})\right]$$

$$\frac{dy}{dt} = f(t,y)$$

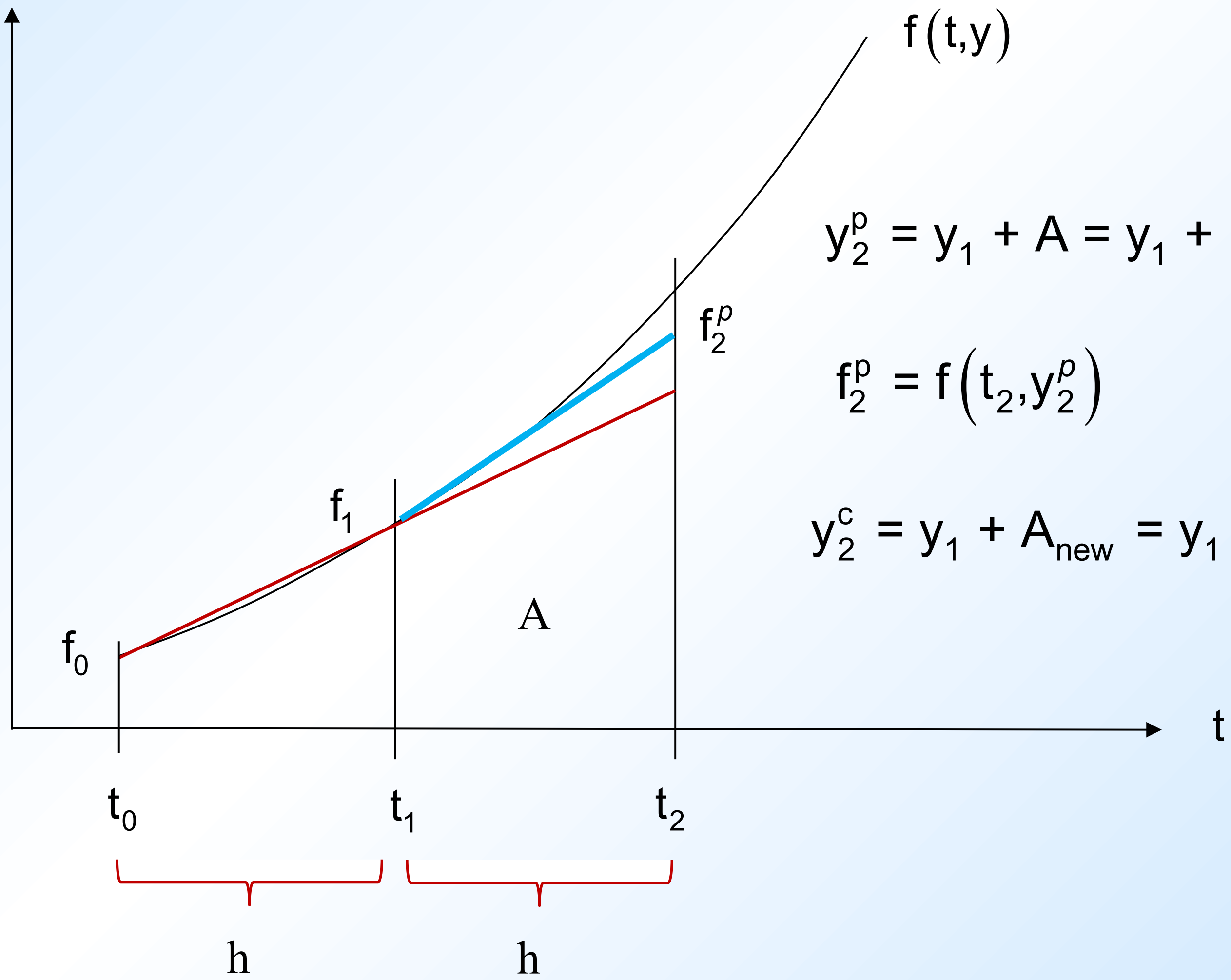


$$y_2 = y_1 + A$$

$$A = h\,f_1 + \frac{h}{2}(f_1 - f_0) = \frac{h}{2}(3\,f_1 - f_0)$$

$$y_2 = y_1 + A = y_1 + \frac{h}{2}(3\,f_1 - f_0)$$

$$\frac{dy}{dt} = f(t,y)$$



$$y_2^p = y_1 + A = y_1 + \frac{h}{2}\left(3\,f_1 - f_0\right)$$

$$f_2^p = f\left(t_2, y_2^p\right)$$

$$y_2^c = y_1 + A_{new} = y_1 + \frac{h}{2}\left(f_1 + f_2^p\right)$$

ADAMS CLOSED FORMULAE

$$y_{i+1} = y_i + h \sum_{k=1}^{n} \alpha_{nk} f_{i-k+1} + O(h^{n+1})$$

Order of the Formula (n)	k = 1 α_{n1}	k = 2 α_{n2}	k = 3 α_{n3}	k = 4 α_{n4}	k = 5 α_{n5}	k = 6 α_{n6}	Local Truncation Error, $O(h^{n+1})$
1	1						$-\frac{1}{2} h^2 f'(\xi)$
2	$\frac{1}{2}$	$\frac{1}{2}$					$-\frac{1}{12} h^3 f''(\xi)$
3	$\frac{5}{12}$	$\frac{8}{12}$	$-\frac{1}{12}$				$-\frac{1}{24} h^4 f'''(\xi)$
4	$\frac{9}{24}$	$\frac{19}{24}$	$-\frac{5}{24}$	$\frac{1}{24}$			$-\frac{19}{720} h^5 f^{(4)}(\xi)$
5	$\frac{251}{720}$	$\frac{646}{720}$	$-\frac{264}{720}$	$\frac{106}{720}$	$-\frac{19}{720}$		$-\frac{27}{1440} h^6 f^{(5)}(\xi)$
6	$\frac{475}{1440}$	$\frac{1427}{1440}$	$-\frac{798}{1440}$	$\frac{482}{1440}$	$-\frac{173}{1440}$	$\frac{27}{1440}$	$-\frac{863}{60480} h^7 f^{(6)}(\xi)$

ADAMS-MOULTON PREDICTOR-CORRECTOR METHODS

Order	Predictor-Corrector Formulae	Local Error
2	$y^p_{n+1} = y_n + \frac{h}{2}\left(3f_n - f_{n-1}\right)$	$\lvert E_{n+1}\rvert \cong \frac{1}{6}\lvert y^p_{n+1} - y^c_{n+1}\rvert$
	$y^c_{n+1} = y_n + \frac{h}{2}\left(f^p_{n+1} + f_n\right)$	
3	$y^p_{n+1} = y_n + \frac{h}{12}\left(23f_n - 16f_{n-1} + 5f_{n-2}\right)$	$\lvert E_{n+1}\rvert \cong \frac{1}{10}\lvert y^p_{n+1} - y^c_{n+1}\rvert$
	$y^c_{n+1} = y_n + \frac{h}{12}\left(5f^p_{n+1} + 8f_n - f_{n-1}\right)$	
4	$y^p_{n+1} = y_n + \frac{h}{24}\left(55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}\right)$	$\lvert E_{n+1}\rvert \cong \frac{19}{270}\lvert y^p_{n+1} - y^c_{n+1}\rvert$
	$y^c_{n+1} = y_n + \frac{h}{24}\left(9f^p_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}\right)$	

Forest Ray Moulton

American Astronomer

1872 - 1952)

Forest Ray Moulton (1872-1952), the youngest of eight children, was born on the family farm between Grand Rapids and Traverse City, Michigan. (The land had been given to his father as part of his bounty for serving in the Union army during the Civil War.) Forest—so named because he was bom in a log cabin in the forest —was educated at Albion College in Michigan, and received a Ph.D. in astronomy from the University of Chicago, in 1899. He is credited, along with his colleague Thomas C. Chamberlain, with formulating the planetesimal hypothesis for the formation of the solar system. During World War I, Moulton did ballistics research for the U.S. Army at Aberdeen Proving Ground, Maryland, and it was during this period that he refined the original work of Adams and Bashforth into what we now know as the Adams-Moulton method for solving initial value problems.

ILL-conditioned ODE: $\dfrac{dy}{dt} = 3y - t^2 \quad , \quad y(0) = \dfrac{2}{27}$

General solution: $y(t) = C e^{3t} + \dfrac{t^2}{3} + \dfrac{2t}{9} + \dfrac{2}{27}$

Particular solution: $y(t) = \dfrac{t^2}{3} + \dfrac{2t}{9} + \dfrac{2}{27}$

Parasitic solution

Example: $\dfrac{du_1}{dx} = 2u_2(x) \quad , \quad u_1(0) = 3 \qquad \dfrac{du_2}{dx} = 2u_1(x) \quad , \quad u_2(0) = -3$

General Solution: $u_1(x) = A e^{2x} + B e^{-2x} \qquad u_2(x) = A e^{2x} - B e^{-2x}$

Apply initial conditions: $A = 0$ and $B = 3$

The component with the positive exponential will dominate with any numerical method

Stiff ODE: Any ODE with a rapidly decreasing transient solution requires an extremely small step size for an accurate solution.

$$\frac{dy(t)}{dt} = \lambda \left(y(t) - g(t) \right) + \frac{dg(t)}{dt} \quad , \quad \lambda << 0 \quad \text{and} \quad g(t) \text{ smooth and slowly varying}$$

Solution: $y(t) = \underbrace{\left[y(0) - g(0) \right] e^{\lambda t}} + g(t)$ h must be very very small

Will soon be insignificant besides g(t)

But, (λ h) will govern the stability. So h must be small as well.

For any resonable h for g(t) will give small (λ h).

For a system of equations $$\frac{dy(t)}{dt} = A\left(y(t) - g(t)\right) + \frac{dg(t)}{dt}$$

the eigenvalues of A correspond to λ. If all the eigenvalues have negative real parts, the solution will converge to g(t) as t goes to infinity.

Example: $$\frac{du}{dt} = 98\,u + 198\,v \qquad \frac{dv}{dt} = -\,99\,u + 199\,v$$

Exact solution:
$$u(t) = 2\,e^{-t} - e^{-100\,t}$$
$$v(t) = -\,e^{-t} + e^{-100\,t}$$

Rapidly decaying terms requiring a very small step size

One answer to stiff ODE's is to use implicit methods: $y_{n+1} = y_n + h\,f\left(t_{n+1}, y_{n+1}\right)$

There exist other methods of solving ODE’s which we will not discuss in detail.

One such method is called Bulirsch-Stoer method. It combines two ideas:

1. Richardson’s extrapolation to the limit; and
2. Rational function (Pade) approximation rather than power (Taylor) series.

See the book «Numerical Recipies».

SYSTEM OF ODE's

Consider:

$$\frac{d\,y(x)}{d\,x} = f\big(x, y(x), z(x)\big) \quad , \quad y(0) = y_0$$

$$\frac{d\,z(x)}{d\,x} = g\big(x, y(x), z(x)\big) \quad , \quad z(0) = z_0$$

All the methods that we have discussed so far are applicable.

Euler's Method:

$$y_{n+1} = y_n + h\,f\big(x_n, y_n, z_n\big)$$

$$z_{n+1} = z_n + h\,g\big(x_n, y_n, z_n\big)$$

SYSTEM OF ODE's

Runge-Kutta Order Four:

$$y_{n+1} = y_n + \frac{h}{6}\left(K_1 + 2\,K_2 + 2\,K_3 + K_4\right)$$

$$z_{n+1} = z_n + \frac{h}{6}\left(L_1 + 2\,L_2 + 2\,L_3 + L_4\right)$$

$$K_1 = f\left(x_n, y_n, z_n\right) \qquad L_1 = g\left(x_n, y_n, z_n\right)$$

$$K_2 = f\left(x_n+\frac{h}{2}, y_n+\frac{h}{2}K_1, z_n+\frac{h}{2}L_1\right) \qquad L_2 = g\left(x_n+\frac{h}{2}, y_n+\frac{h}{2}K_1, z_n+\frac{h}{2}L_1\right)$$

$$K_3 = f\left(x_n+\frac{h}{2}, y_n+\frac{h}{2}K_2, z_n+\frac{h}{2}L_2\right) \qquad L_3 = g\left(x_n+\frac{h}{2}, y_n+\frac{h}{2}K_2, z_n+\frac{h}{2}L_2\right)$$

$$K_4 = f\left(x_{n+1}, y_n+h\,K_3, z_n+h\,L_3\right) \qquad L_4 = g\left(x_{n+1}, y_n+h\,K_3, z_n+h\,L_3\right)$$

HIGHER-ORDER ODE's

Second-order ODE: $$\frac{d^2y(x)}{d\,x^2} = f\left(x, y(x), y'(x)\right)$$

Two conditions must be specified for the solution:

At $x = 0$ $\quad y(0) = y_0$ and

$$\left.\frac{d\,y}{d\,x}\right|_{x=0} = y'(0)$$

} Initial-value problem

At $x = 0$ $\quad y(0) = y_0$ and

At $x = L$ $\quad y(L) = y_L$

} Boundary-value problem

Note that derivative conditions may also be specified at the boundaries

SECOND-ORDER ODE - INITIAL VALUE PROBLEM

$$\frac{d^2y(x)}{d\,x^2} = f\big(x,\, y(x),\, y'(x)\big)$$

At $x = 0$ $\quad y(0) = y_0$ and

$$\left.\frac{d\,y}{d\,x}\right|_{x=0} = y'(0)$$

Define

$$\frac{dy(x)}{d\,x} = z(x) = g(x, y, z) \quad , \quad z(0) = z_0$$

$$\frac{d\,z(x)}{d\,x} = f\big(x,\, y(x),\, z(x)\big) \quad , \quad y(0) = y_0$$

All the methods discussed so far are applicable.

SECOND-ORDER ODE - INITIAL VALUE PROBLEM

Given:	$y'' = -y$	$y(0) = 0$	$y'(0) = 1$
Exact solution:	$y(t) = \sin(t)$	$y'(t) = \cos(t)$	

t	0	0.1	0.2	0.3	0.4	0.5	1.0
y_{exact}	0	0.0998	0.1987	0.2955	0.3894	0.4794	0.8415
y_{Euler}	0	0.100	0.200	0.299	0.396	0.440	0.940
y'_{exact}	1.0	0.995	0.980	0.995	0.921	0.878	0.540
y'_{Euler}	1.0	1.0	0.990	0.970	0.940	0.900	0.660

BOUNDARY-VALUE PROBLEMS

Second-order ODE: $$\frac{d^2y(x)}{d\,x^2} = f\left(x,\, y(x),\, y'(x)\right)$$

At $x = 0$ $y(0) = y_0$ and

At $x = L$ $y(L) = y_L$

} Boundary-value problem

Note that derivative conditions may also be specified at the boundaries

There are two methods of solution: Matrix Method - f is a linear function of y and all derivatives of y

Shooting Method - A trial-error solution

Matrix Method

Given: $$\frac{d^2y(x)}{d\,x^2} = f\big(x,\ y(x),\ y'(x)\big) \quad , \quad y(a) = \alpha \quad , \quad y(b) = \beta$$

Select a set of equally-spaced points, x_0, x_1, ..., x_n, x_{n+1}, on the interval [a,b]

$$x_i = a + i\,h \quad , \quad h = \frac{b - a}{n + 1} \quad , \quad 0 \le i \le n + 1$$

Approximate the derivatives using standart central-difference formula

$$y'(x) \cong \frac{y(x+h) - y(x-h)}{2\,h} = \frac{y_{i+1} - y_{i-1}}{2\,h}$$

$$y''(x) \cong \frac{y(x+h) - 2\,y(x) + y(x-h)}{h^2} = \frac{y_{i+1} - 2\,y_i + y_{i-1}}{h^2}$$

The problem becomes:

$$y_0 = \alpha \quad \text{(given)}$$

$$\frac{y_{i-1} - 2\,y_i + y_{i+1}}{h^2} = f\left(x_i,\, y_i,\, \frac{y_{i+1} - y_{i-1}}{2\,h} \right) \quad , \quad 1 \le i \le n$$

$$y = \beta \quad \text{(given)}$$

This is usually a non-linear system of equations in n unkowns, y_1, y_2, ..., y_n.

The solution of such a sytem of non-linear equations is seldom easy.

IF the **"f" function is linear**, only then the solution is easier because we can form a matrix:

Example

Solve the following second-order ODE with the given boundary conditions:

$$- 2 \frac{d^2y}{dx^2} + y = e^{-0.2x} \quad , \quad y(0) = 1 \quad , \quad \left.\frac{dy}{dx}\right|_{x=1} = - y$$

Divide the solution domain into eight subintervals, and use the central difference approximation for all the derivatives in the given ODE and the boundary conditions. Compare the numerical solution with the exact solution:

Exact solution: $$y = - 0.2108\, e^{x/\sqrt{2}} + 0.1238\, e^{-x/\sqrt{2}} + \frac{e^{-0.2x}}{0.92}$$

Note: When there is an exact solution, numerical solution becomes unnecessary.

Given ODE: $- 2 \dfrac{d^2y}{dx^2} + y = e^{-0.2x}$

Corresponding finite difference equation:

$$- 2 \left(\frac{y_{i-1} - 2 y_i + y_{i+1}}{h^2} \right) + y_i = e^{-0.2 x_i} \quad , \quad i = 1, 2, ..., 7$$

Re-arrange: $- 2 y_{i-1} + \left(4 + h^2\right) y_i - 2 y_{i+1} = h^2 e^{-0.2 x_i} \quad , \quad i = 1, 2, ..., 7$

Boundary conditions: $y_0 = 1$ at $x = 0$

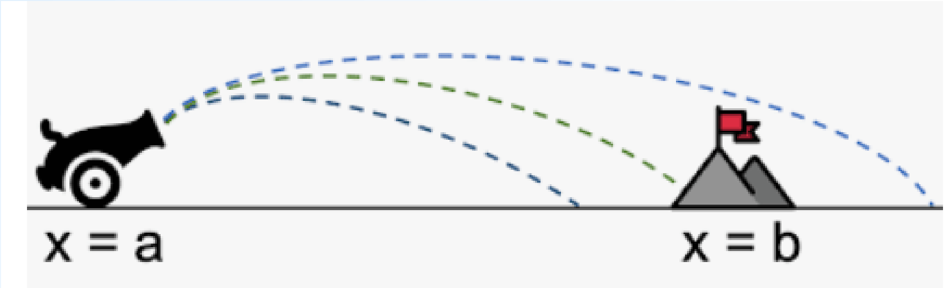
$$\left.\frac{dy}{dx}\right|_{x=1} \cong \frac{y_9 - y_7}{2h} = - y_8 \quad => \quad y_9 = y_7 - 2 h y_8$$

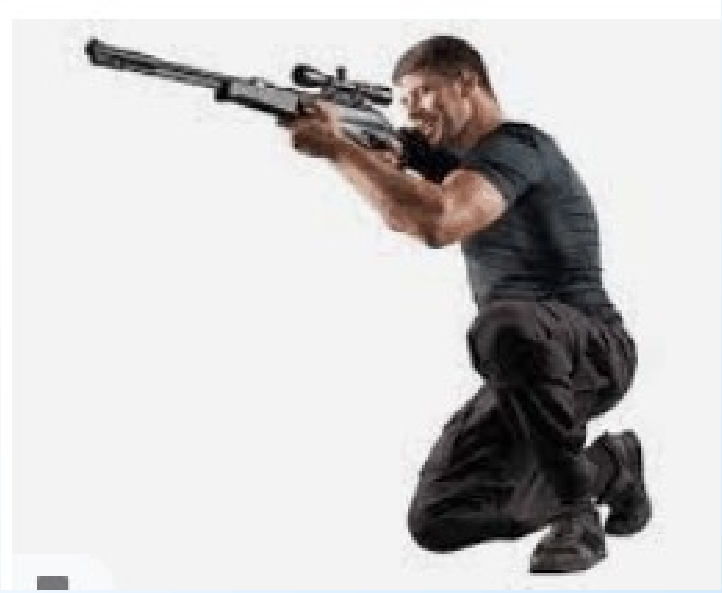
$$
\begin{pmatrix}
4+h^2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 4+h^2 & -2 & 0 & 0 & 0 & 0 & 0 \\
0 & -2 & 4+h^2 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & 4+h^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2 & 4+h^2 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & -2 & 4+h^2 & -2 & 0 \\
0 & 0 & 0 & 0 & 0 & -2 & 4+h^2 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & -4 & 4+4h+h^2
\end{pmatrix}
\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \end{pmatrix}
=
\begin{pmatrix}
2+h^2\ e^{-0.2x_1} \\
h^2\ e^{-0.2x_2} \\
h^2\ e^{-0.2x_3} \\
h^2\ e^{-0.2x_4} \\
h^2\ e^{-0.2x_5} \\
h^2\ e^{-0.2x_6} \\
h^2\ e^{-0.2x_7} \\
h^2\ e^{-0.2x_8}
\end{pmatrix}
$$

where $h = 1/8$ and $x_i = i\,h$ for $i = 1, 2, \ldots, 8$

	Matrix	Exact
y_0 =	1.00000	1.00000
y_1 =	0.94323	0.94317
y_2 =	0.88620	0.88612
y_3 =	0.82867	0.82857
y_4 =	0.77036	0.77025
y_5 =	0.71101	0.71087
y_6 =	0.65031	0.65014
y_7 =	0.58797	0.58778
y_8 =	0.52366	0.52344

Shoothing to hit the target by adjusting the aim (slope) of the gun or a rifle.

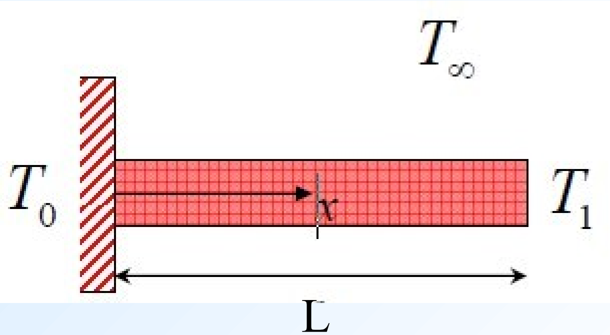




Shooting Method

- Applicable to both linear & non-linear Boundary Value (BV) problems.
- Easy to implement
- No guarantee of convergence
- Approach:
  - Convert a BV problem into an initial value problem
  - Solve the resulting problem iteratively (trial & error)
  - Linear ODEs allow a quick linear interpolation
  - Non-linear ODEs will require an iterative approach similar to our root finding techniques.

Cooling Fin Example



$$\frac{d^2T}{dx^2} - \frac{h\,P}{k\,A}\left(T - T_\infty\right) = 0$$

$$T(x = 0) = T_0$$

$$T(x = L) = T_L$$

h = heat transfer coefficient

k = thermal conductivity

P = perimeter of fin

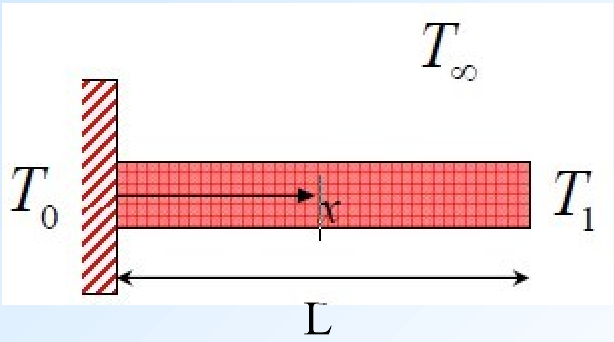
A = cross sectional area of fin

T_∞ = ambient temperature

Analytical solution:

$$m^2 = \frac{h\,P}{k\,A} \qquad \theta(x) = T(x) - T_\infty$$

$$\frac{d^2\theta}{dx^2} - m^2\,\theta = 0$$



$$\frac{d^2\theta}{dx^2} - m^2\,\theta = 0$$

$$\theta(x) = C_1\, e^{m\,x} + C_2\, e^{-m\,x}$$

$$\theta(x) = T(x) - T_\infty$$

$$m^2 = \frac{h\,P}{k\,A}$$

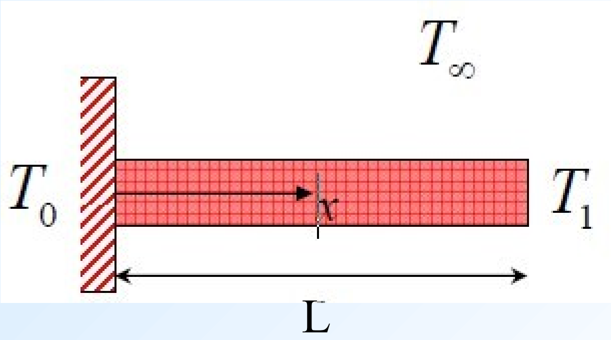
Boundary Conditions:

$T(0) = T_0 \quad => \quad \theta(0) = T_0 - T_\infty = \theta_0$

$T(L) = T_L \quad => \quad \theta(L) = T_L - T_\infty = \theta_L$

Apply BC's and solve for C_1 and C_2

$$\frac{\theta(x)}{\theta_0} = \frac{\left(\theta_L/\theta_0\right)\,\sinh(m\,x) + \sinh\left(m\,(L - x)\right)}{\sinh(m\,L)}$$



$$\frac{d^2T}{dx^2} - \frac{h\,P}{k\,A}\left(T - T_\infty\right) = 0$$

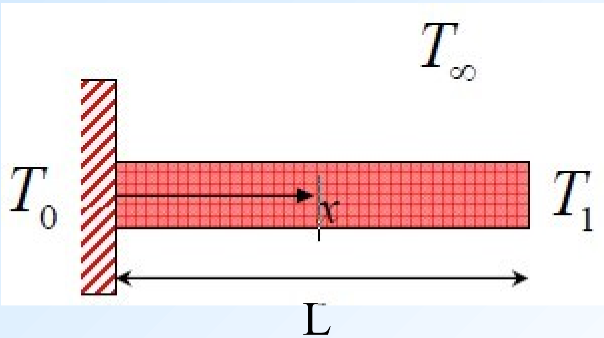
BC's: $T(0) = T_0$

$T(L) = T_L$

1. Re-write as two first-order ODE's: $\frac{dT}{dx} = z \quad , \quad T(0) = T_0$

$$\frac{dz}{dx} = \frac{h\,P}{k\,A}(T - T_\infty) \quad , \quad z(0) = ?$$

2. We need an initial value for z. **Guess z_1**



3. Integrate the two equations using RK4 and z_1; this will yield a solution at x = L

4. Integrate the two equations again using a 2nd guess for $z(0) = z_2$.

5. Linearly interpolate the *z* results to obtain the correct initial condition

(Note: this only works for Linear ODEs.)

Example: $$z_3 = z_2 + \frac{z_1 - z_2}{T_1 - T_2}\left(T_L - T_2\right)$$