SOLUTION OF ENGINEERING PROBLEMS

Given	A real physical problem such as design and manufacture of a robot arm, heat exchanger, etc.
Determine	Length, area, volume, material, velocity, acceleration, forces, pressure, temperature, flow rates of mass and energy, etc.
Mathematical Formulation	Use simplifying assumptions => **Physical Modeling**
	Make force, momentum, mass, energy balances
	Obtain mathematical equations in the form of differential, integral, integro-differential, algebraic, set of algebraic, etc.
	Make further simplifications on the mathematical formulae => **Mathematical Modeling**

SOLUTION OF ENGINEERING PROBLEMS

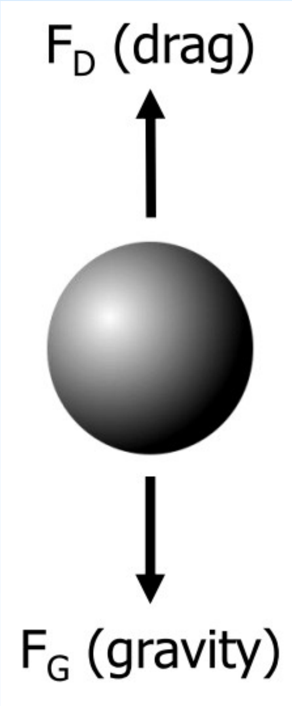
Solve Mathematical Formulae	• Analytically (exactly) • **Numerically (approximately), using methods such as finite differences, finite elements, spectral methods, etc., using *computers and programming***

Report Errors and Interpret Results	With respect to the solution technique With respect to the simplifications in the mathematical formulae With respect to the simplifications in the physical model

See the pdf file «Engineering Mathematicians» on the Moodle.

Example

$$m\,a = m\,\frac{dv(t)}{dt} = F_G - F_D = m\,g - F_D \qquad v(t) = ?$$



Physical model: Perfect sphere

Mathematical model: Assume F_D is proportinal to velocity

$$F_D = C_D\,v(t)$$

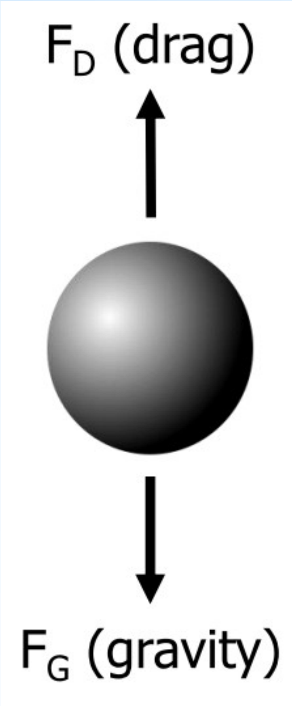
$$\frac{dv(t)}{dt} = \frac{m\,g - C_D\,v(t)}{m}$$

Analytical (exact) solution:

$$v(t) = \frac{g\,m}{C_D}\left[1 - \exp\left(\frac{C_D}{m}\,t\right)\right]$$

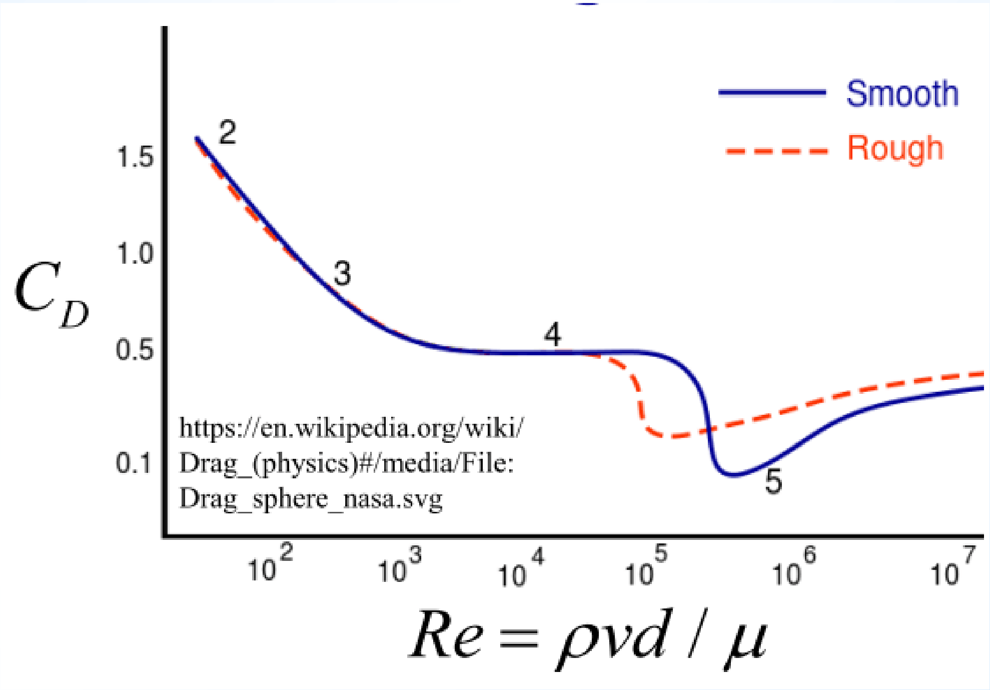
Example

Relax the mathematical model:



$$F_D = \frac{1}{2} \rho \left[v(t)\right]^2 C_D A_P$$

$$\frac{dv(t)}{dt} = \frac{m\,g - F_D}{m}$$



v(t) = ?

COMPUTERS AND PROGRAMMING

A computer program (code) a set (list) of commands (operations) to be executed.

Commands:

- Input and Output
- Defining variables
- Execution of mathematical operations
- Controlling the order in which commands are executed
- Repeating sections of the program
- Creating figures and charts

Algorithm is the plan of implementation on a computer.

Problem: Find the real roots of the equation: $a\,x^2 + b\,x + c = 0$

$$x_1 = \frac{-b + \sqrt{b - 4ac}}{2a} \qquad x_2 = \frac{-b - \sqrt{b - 4ac}}{2a}$$

Algorithm:

- Start and read in the given values for a, b, and c
- Calculate the value of the discriminant $D = b^2 - 4ac$
- If $D \geq 0$, calculate the two real roots using the equations, and write them
- If $D = 0$, calculate the root $x = -b/2a$, write and display “Single root”
- If $D < 0$, display “No real root”
- Stop

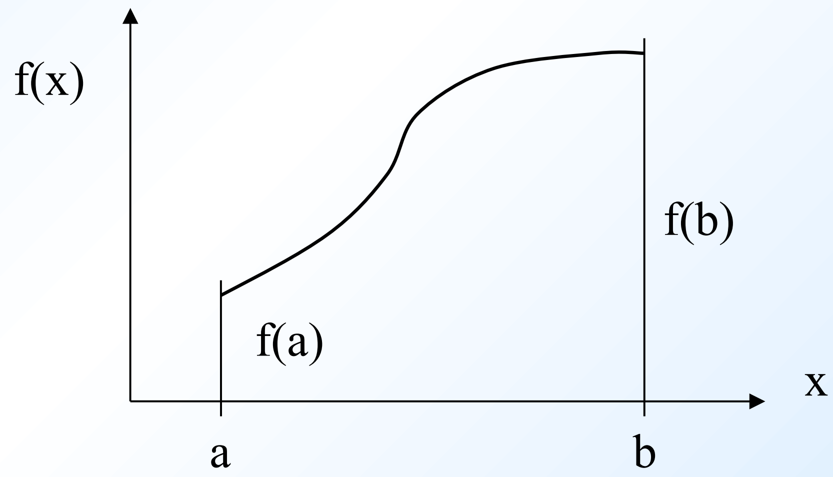
COMPUTATIONAL PROCEDURE

Frequently, several methods are available for the numerical solution of a given mathematical problem. Few relevant criteria for selection of a method:

- Accuracy
- Efficiency
- Numerical stability
- Programming simplicity
- Versatility
- Computer storage requirements
- Interfacing with available software
- Previous experience with a given method

REVIEW OF CALCULUS

Given: f(x), a real-valued continuous function



Open and Closed Intervals:

Open : (a , b) : in a to b

Closed : [a , b] : in and on a to b

Continuous Function

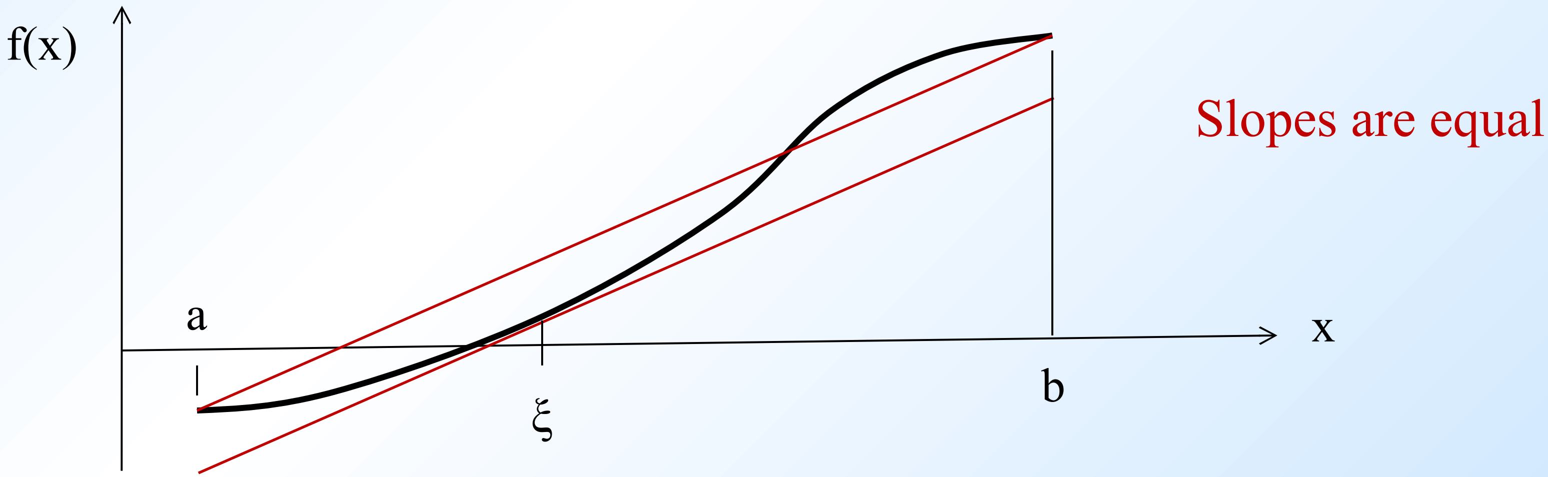
If a real valued function, f (x), is defined in (a , b) it is said to be continuous on (a , b) at a point x_0 if, for every $\varepsilon > 0$, there exists a positive, non-zero δ such that $|x - x_0| < \delta$, $|f(x) - f(x_0)| < \varepsilon$

If a function is continuous for all x values in an interval, it is said to be continuous on the interval.

Mean-Value Theorem for Derivatives

When f(x) is continuous on [a , b] and f '(x) exists for all interior points, then at some ξ

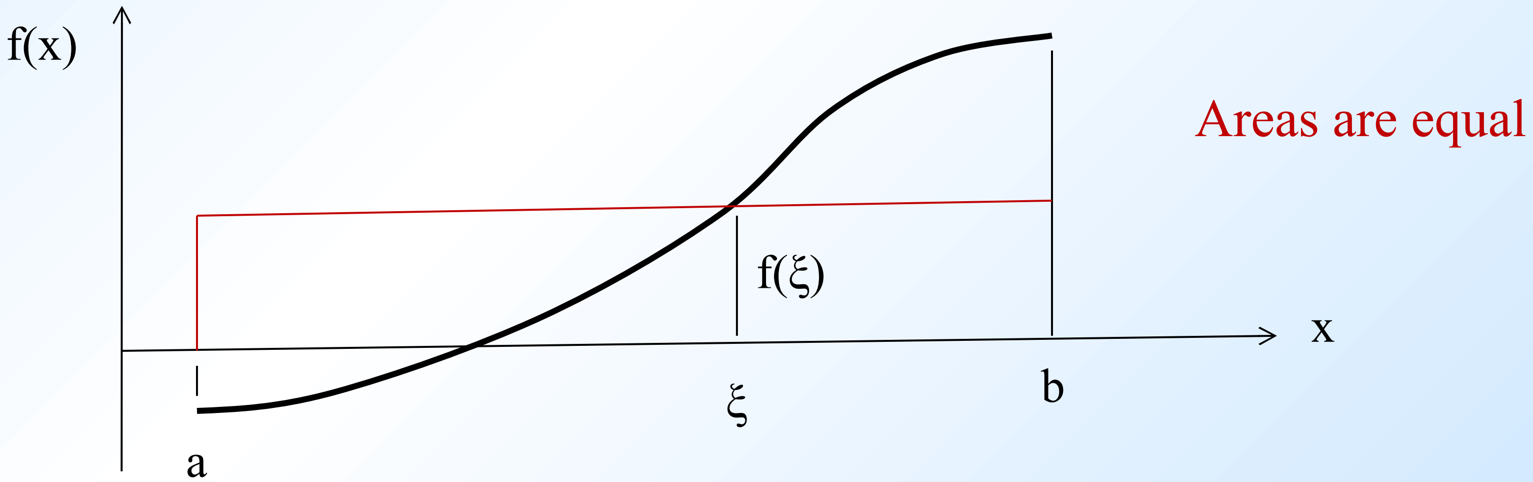
$$f'(\xi) = \frac{f(b) - f(a)}{b - a} \quad , \quad a < \xi < b$$



Mean-Value Theorem for Integrals

If f(x) is continuous and integrable on [a , b] then

$$\int_a^b f(x)\,dx = (b-a)\,f(\xi) \qquad , \quad a < \xi < b$$



Similarly, if f (x) and g (x) are continuous and integrable on [a , b], and if g (x) does not change sign on [a , b], then

$$\int_a^b f(x)\, g(x)\, dx = f(\xi) \int_a^b g(x)\, dx \qquad , \quad a < \xi < b$$

Geometrical interpretation is that the right-hand-side represents a volume.

This corollary is what we will use most of the time.

Taylor Series

If a function f(x) and all its derivatives exist at a point $x = x_0$, the function f(x) can be represented by an infinite power series in powers of $(x - x_0)$.

$$f(x) = f(x_0) + (x - x_0) f'(x_0) + \frac{(x - x_0)^2}{2!} f''(x_0) + \dots + \frac{(x - x_0)^n}{n!} f^{(n)}(x_0) + R$$

$$R = \frac{(x - x_0)^{n+1}}{(n+1)!} f^{(n+1)}(\xi)$$

Remember radius of convergence

Set $x - x_0 = \Delta x$:

$$f(x) = f(x_0) + \Delta x f'(x_0) + \frac{(\Delta x)^2}{2!} f''(x_0) + \dots + \frac{(\Delta x)^n}{n!} f^{(n)}(x_0) + R$$

$$R = \frac{(\Delta x)^{n+1}}{(n+1)!} f^{(n+1)}(\xi) \quad , \quad x_0 < \xi < x$$

In another form, substitute $x - x_0 = \Delta x = h$

$$f(x) = f(x_0) + h f'(x_0) + \frac{h^2}{2!} f''(x_0) + \dots + \frac{h^n}{n!} f^{(n)}(x_0) + R$$

$$R = \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(\xi) \quad , \quad x_0 < \xi < x$$

In another form, substitute $x = x_0 + h$

$$f(x_0 + h) = f(x_0) + h f'(x_0) + \frac{h^2}{2!} f''(x_0) + \dots + \frac{h^n}{n!} f^{(n)}(x_0) + R$$

$$R = \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(\xi) \quad , \quad x_0 < \xi < x$$

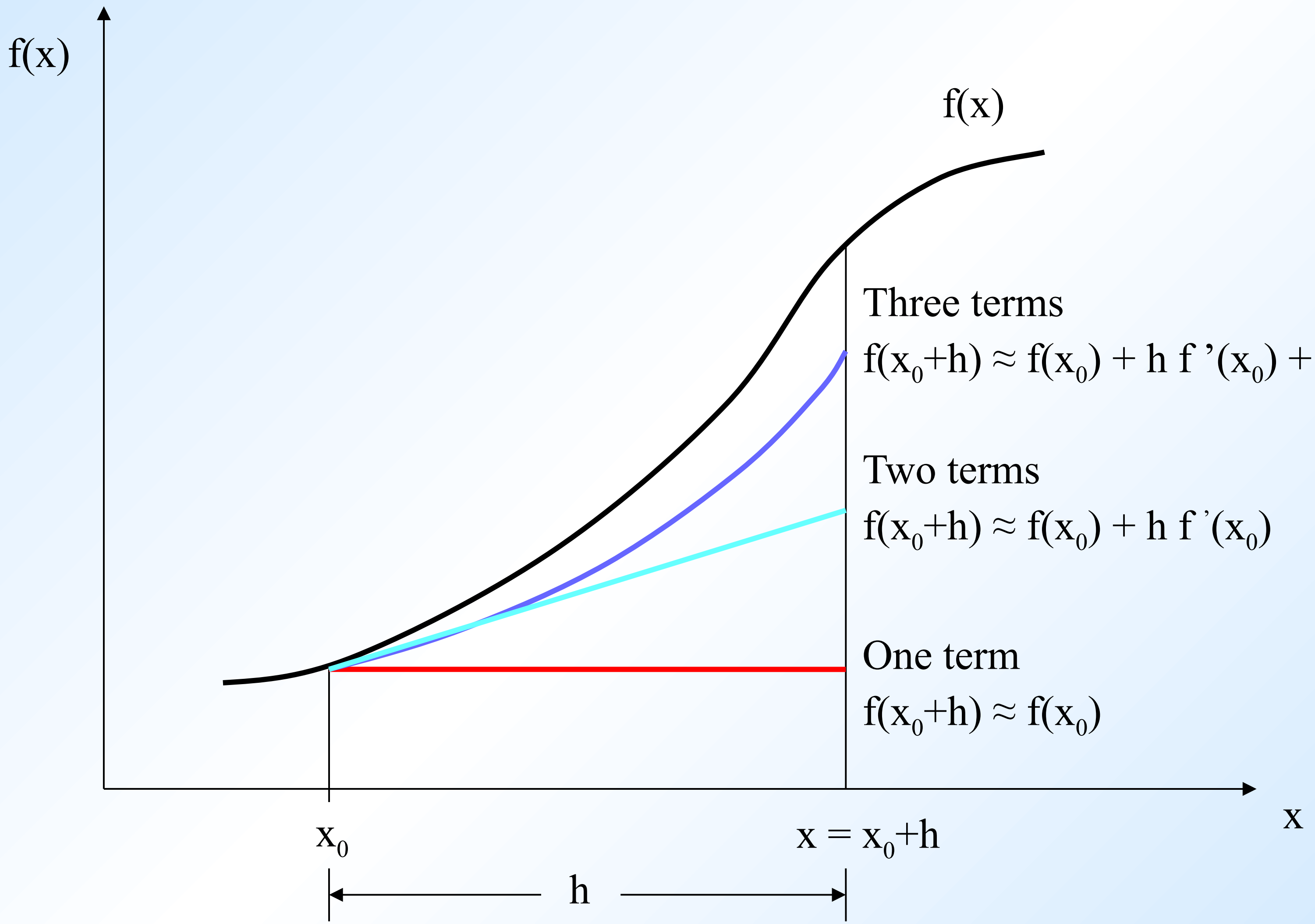
Taylor Series and Mean Value Theorem or Integrals

$$\int_a^b f(x)\, g(x)\, dx = f(\xi) \int_a^b g(x)\, dx \qquad , \quad a < \xi < b$$

$$R = \int_{x_0}^{x} \frac{(x - t)^n}{n!} f^{(n+1)}(t)\, dt = f^{(n+1)}(\xi) \int_{x_0}^{x} \frac{(x - t)^n}{n!}\, dt$$

$$R = f^{(n+1)}(\xi) \frac{(x - x_0)^{n+1}}{(n+1)!} \qquad , \quad x_0 < \xi < x$$

Approximations with Different Number of Terms



Example 1

Consider the function: $f(x) = \dfrac{1}{1 - x} \quad , \quad x \neq 1$

Taylor Series expansion around a chosen point x_0:

$$f(x) = f(x_0) + (x - x_0) f'(x_0) + \frac{(x - x_0)^2}{2!} f''(x_0) + \frac{(x - x_0)^3}{3!} f'''(x_0) + \ldots$$

$$\ldots + \frac{(x - x_0)^n}{n!} f^{(n)}(x_0) + \frac{(x - x_0)^{n+1}}{(n + 1)!} f^{(n+1)}(\xi) \quad , \quad x_0 < \xi < x$$

Derivatives: $f'(x_0) = \dfrac{1}{(1 - x_0)^2}$ $\quad$ $f''(x_0) = \dfrac{2}{(1 - x_0)^3}$

Approximations to f(x):

One term,
First Order

$$f(x) \cong \frac{1}{1-x_0} \qquad \text{Error} = (x - x_0)\, f'(\xi)$$

Two terms,
Second Order

$$f(x) \cong \frac{1}{1-x_0} + \frac{(x-x_0)}{(1-x_0)^2} \qquad \text{Error} = \frac{(x-x_0)^2}{2!} f''(\xi)$$

Three terms,
Third Order

$$f(x) \cong \frac{1}{1-x_0} + \frac{(x-x_0)}{(1-x_0)^2} + \frac{(x-x_0)^2}{(1-x_0)^3} \qquad \text{Error} = \frac{(x-x_0)^3}{3!} f'''(\xi)$$

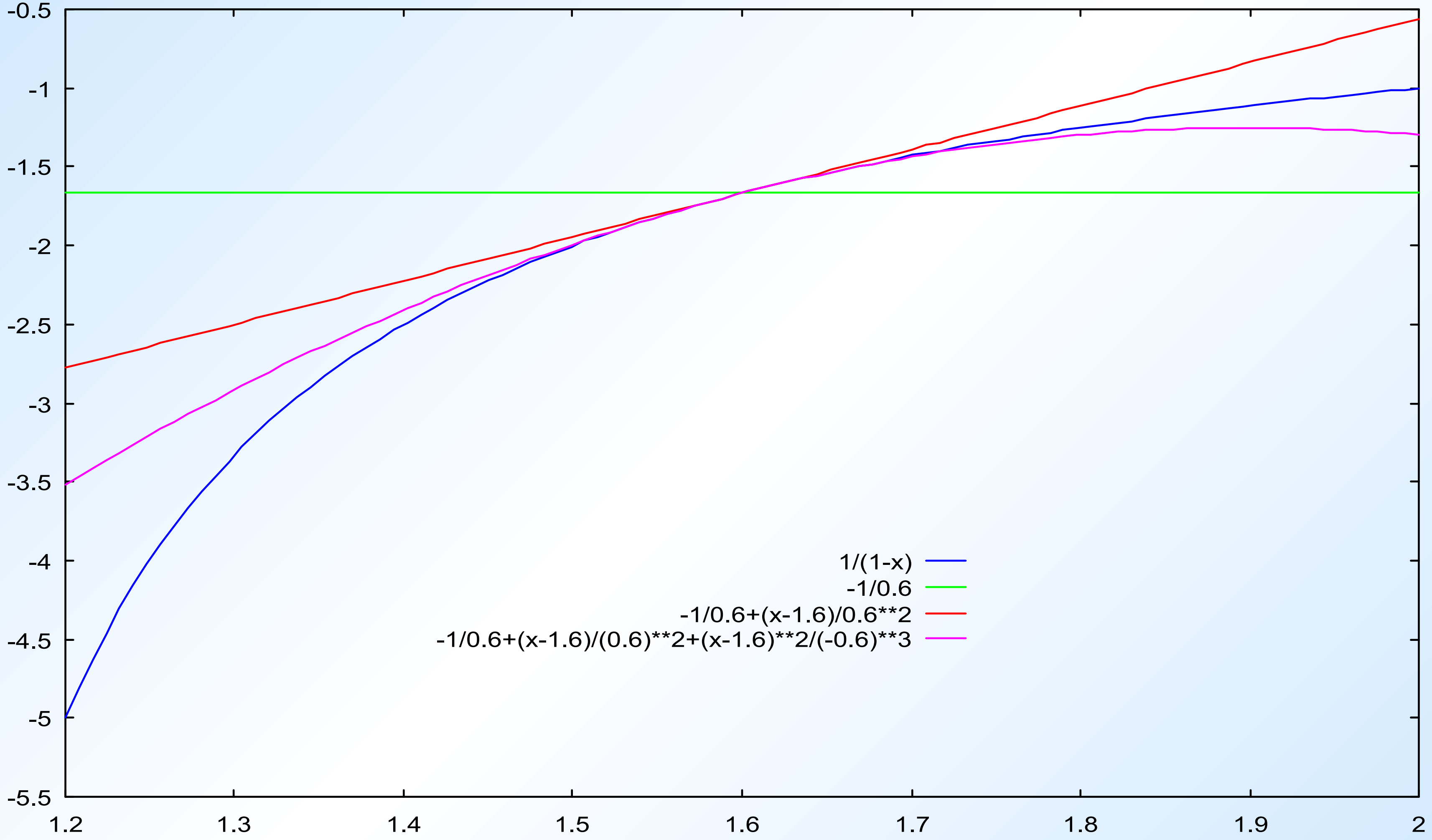
Example 2 If x_0 = 1.6. Estimate f(2) using two terms in the Taylor series expansion; Find the true and estimated errors

True Error: $$\left|E_{true}(x)\right| = \left|f(x) - P_1(x)\right|$$

True Relative Error: $$\left|E_{true}(x)\right| = \left|\frac{f(x) - P_1(x)}{f(x)}\right| * 100 \quad \text{in percent}$$

Estimated Error: $$\left|E_{est}(x)\right| = \left|\frac{(x - x_0)^2}{2!} f''(\xi)\right|$$

x	f(x)	Estimation with two terms	True Error		Estimated Error	
					$\xi = 1.6$	$\xi = 2$
2	-1.0	-0.55556	0.44444	44.4 %	Max 0.74	Min 0.16



Error Calculation: Find the error at $x = 2$

Example 3

If one applies integration by parts to the integral

$$y_n = \int_0^1 x\, e^x \, dx$$

defining y_{n+1}, the result is the recurrence relation

$$y_{n+1} = e - (n + 1)\, y_n$$

Starting with $y_0 = e - 1$, and $y_1 = 1$, generate y_2, $y_{3}, \ldots, y_{15}$ using a compiler with single-precision (7 digit accuracy) and double-precision (15 digit accuracy) calculations. Comment on the results.

n	$y_{n,single}$
0	1.71828183
1	1.00000000
2	0.71828183
3	0.56343634
4	0.46453647
5	0.39559948
6	0.34468495
7	0.30548718
8	0.27438439
9	0.24882232
10	0.23005863
11	0.18763690
12	0.46663903
13	-3.34802556
14	49.59063967
15	-741.14131322

Why is this happening?

n	$y_{n,single}$	$y_{n,double}$
0	1.71828183	1.71828183
1	1.00000000	1.00000000
2	0.71828183	0.71828183
3	0.56343634	0.56343634
4	0.46453647	0.46453646
5	0.39559948	0.39559955
6	0.34468495	0.34468454
7	0.30548718	0.30549004
8	0.27438439	0.27436153
9	0.24882232	0.24902803
10	0.23005863	0.22800152
11	0.18763690	0.21026516
12	0.46663903	0.19509991
13	-3.34802556	0.18198305
14	49.59063967	0.17051906
15	-741.14131322	0.16049585

Use double precision all the time

Example 4

The "machine epsilon", ε, of a computer is the smallest positive number in the form $\varepsilon = 2^{-k}$ such that $1.0 + \varepsilon \neq 1.0$. It is also called the unit of round-off error of the machine.

Use the following algorithm to compute an approximate value of ε for the computer that you intend to use in this course.

```
input s == 1.0
for k = 1, 2, … 100 do
      s == 0.5 * s
      t == s + 1.0
      if t ≤ 1.0 then
          s == 2.0 * s
          output k-1 , s
          stop
       end if
end
```

Example 5

Expand the function $f(x) = \sin\left(\frac{\pi x}{2}\right)$ in Taylor series around the point $x_0 = 0$ (MacLaurin series). Find the expression for the remainder, and estimate the number of terms that would be needed to guarantee six significant digit accuracy for all x in the interval [-1,+1].

Six significant digit accuracy means that the error should be less than or equal to $0.5\ 10^{-6}$.

$$f(x) = \sin\left(\frac{\pi x}{2}\right) = f(0) + x\, f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots + \frac{x^n}{n!} f^{(n)}(0) + \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(\xi)$$

$$f(x) = \sin\left(\frac{\pi x}{2}\right) = \frac{\pi}{2} x - \left(\frac{\pi}{2}\right)^3 \frac{x^3}{3!} + \left(\frac{\pi}{2}\right)^5 \frac{x^5}{5!} - \left(\frac{\pi}{2}\right)^7 \frac{x^7}{7!} + \ldots + R$$

$$|R| = \left(\frac{\pi}{2}\right)^{2n+1} \frac{x^{2n+1}}{(2\,n+1)!} \sin\left(\frac{\pi}{2}\xi\right) \quad , \quad -1 < \xi < 1 \qquad n = 1, 2, 3, \ldots$$

The maximum error is expected to occur at $x = 1$ and $\xi = 1$.

For 6 significant digit accuracy, R should be less than $0.5\ 10^{-6}$.

$$|R|_{max} = \left(\frac{\pi}{2}\right)^{2n+1} \frac{1}{(2\,n+1)!} < 0.5\ 10^{-6} \qquad \text{Find n.}$$

No of non-zero terms	R_{max}
1	$6.45964\ 10^{-1}$
2	$7.96926\ 10^{-2}$
3	$4.68175\ 10^{-3}$
4	$1.60441\ 10^{-4}$
5	$3.59884\ 10^{-6}$
6	$5.69217\ 10^{-8}$

n = 6 is the required number of non-zero terms

$$e^x = 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + ... + \frac{1}{n!} x^n + \frac{1}{(n+1)!} x^{n+1} e^{\xi_x}$$

$$= \sum_{k=0}^{n} \frac{1}{k!} x^k + R_n(x)$$

$$\sin(x) = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + ... + \frac{(-1)^n}{(2\,n+1)!} x^{2n+1} + \frac{(-1)^{n+1}}{(2\,n+3)!} x^{2n+3} \cos(\xi_x)$$

$$= \sum_{k=0}^{n} \frac{1}{k!} x^k + R_n(x)$$

$$\cos(x) = 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 + ... + \frac{(-1)^n}{(2\,n)!} x^{2n} + \frac{(-1)^{n+1}}{(2\,n+2)!} x^{2n+2} \cos(\xi_x)$$

$$= \sum_{k=0}^{n} \frac{1}{(2k)!} x^{2k} + R_n(x)$$

Taylor Series expansion of a function of two variables, f(x,y), around (x_0,y_0)

$$f(x,y) = f(x_0,y_0) + (x - x_0)\left.\frac{\partial f}{\partial x}\right|_{x_0,y_0} + (y - y_0)\left.\frac{\partial f}{\partial y}\right|_{x_0,y_0} +$$

$$\frac{1}{2\,!}\left[(x - x_0)^2\left.\frac{\partial^2 f}{\partial x^2}\right|_{x_0,y_0} + (y - y_0)^2\left.\frac{\partial^2 f}{\partial y^2}\right|_{x_0,y_0} + 2\,(x - x_0)\,(y - y_0)\left.\frac{\partial^2 f}{\partial x\,\partial y}\right|_{x_0,y_0}\right]$$

$$+ \ldots$$

Note that these expansions are valid within a radius of convergence, r.

For one-variable case: $\left|(x - x_0)\right| < r$

Example 6

The Stefan-Boltzmann law states that the radiative energy, Φ, emitted from an object is given by the relation

$$\Phi = A\, \varepsilon\, \sigma\, T^4$$

where Φ is in Watts, A is the surface area in m^2, ε is emissivity, σ is Stephan-Boltzmann constant equal to 5.67 10^{-8} $W/m^2.K^4$, and T is absolute temperature in K. Using the general error estimator of the Taylor series, estimate the error in Φ for a copper sphere with radius r = 0.1 ± 0.01 m, emissivity ε = 0.5 ± 0.05, and temperature T = 500 ± 20 K. Compare your result with the actual error.

$$\Phi(r, \varepsilon, T) = 4\, \pi\, r^2\, \varepsilon\, \sigma\, T^4 \quad \Rightarrow \quad \partial\phi = \left|\frac{\partial\phi}{\partial r}\right| \partial r + \left|\frac{\partial\phi}{\partial\varepsilon}\right| \partial\varepsilon + \left|\frac{\partial\phi}{\partial T}\right| \partial T$$

Solution of Example 6

Partial derivatives: $\frac{\partial\Phi}{\partial r} = 8\,\pi\, r\, \varepsilon\, \sigma\, T^4$ $\frac{\partial\Phi}{\partial \varepsilon} = 4\,\pi\, r^2\, \sigma\, T^4$ $\frac{\partial\Phi}{\partial T} = 16\,\pi\, r^2\, \varepsilon\, \sigma\, T^3$

Substitute the given values: Φ(r, ε, T) = 222.66 W

$\frac{\partial\Phi}{\partial \varepsilon} = 445.3$ $\frac{\partial\Phi}{\partial r} = 4453$ $\frac{\partial\Phi}{\partial T} = 1.781$

$$\Delta\phi = \left|\frac{\partial\phi}{\partial r}\right| \Delta r + \left|\frac{\partial\phi}{\partial \varepsilon}\right| \Delta\varepsilon + \left|\frac{\partial\phi}{\partial T}\right| \Delta T = (4453)\,(0.01) + 445.3)\,(0.05) + (1.781)\,(20)$$

= 102.42 W

Therefore: Φ(r, ε, T) = 222.66 ∓ 102.42

This is almost 50 % error.

APPROXIMATING FUNCTIONS

Objectives:

- Replace a complicated function f(x) with a simpler function g(x) which is easier to manipulate, i.e. differentiate, integrate, etc. (Note that transcendental functions, such as ln(x), sin(x), erf(x), etc., cannot be evaluated by strictly arithmetic operations without first finding approximating, simpler functions such as a finite power series.)
- Interpolate in tables of functional values

Power series:

$$g(x) = P_n(x) = \sum_{i=0}^{n} a_i \, x^i$$

Fourier Functions:

$$g(x) = a_0 + \sum_{k=1}^{n} a_k \cos(k\,x) + \sum_{k=1}^{n} a_k \sin(k\,x)$$

Exponentials

$$g(x) = \sum_{i=0}^{n} a_i\, e^{b_i x}$$

Rational Functions

$$g(x) = \frac{P_n(x)}{P_m(x)} = \frac{\sum_{i=0}^{n} a_i\, x^i}{\sum_{i=0}^{m} c_i\, x^i}$$

Padé approximation

SENSITIVITY and CONDITION

A problem is *insensitive*, or *well-conditioned*, if given relative change in the input causes commensurate relative change in the solution.

A problem is *sensitive*, or *ill-conditioned*, if relative change in solution can be much larger than that in input data.

$$\text{Condition Number} = \frac{|\text{Relative Change in solution}|}{|\text{Relative change in input data}|} = \frac{\left[f(\bar{x}) - f(x) \right] / f(x)}{(\bar{x} - x) / x}$$

Problem is *sensitive* or *ill-conditioned* if Condition Number >> 1

Example: Sensitivity

Consider the problem of computing cosine function for arguments near π /2.

Let x ≈ π /2 and let h be a small perturbation to x.

Then, the error in cos(x + h) is given by

$$\text{Absolute Error} = \cos(x + h) - \cos(x) \approx -h \sin(x) \approx -h$$

$$\text{Relative Error} \approx -h \tan(x) \approx \infty$$

So, small changes in x near π /2 cause large relative changes in cos(x) regardless of method for computing it.

Example: $\cos(1.57079) = 0.63267949\ 10^{-5}$

$\cos(1.57078) = 1.63267949\ 10^{-5}$

Relative change in output is about a quarter million times greater than relative change in input.

Example: Condition Number for a Function

When a function, f, is evaluated for an approximate input argument $\Delta x = x + h$ instead of true input value x:

Error in the function = $f(x + h) - f(x) \approx \Delta x\, f'(x) = h\, f'(x)$

$$\text{Relative error in the function} = \frac{f(x+h) - f(x)}{f(x)} = \frac{\Delta f(x)}{f(x)} \approx h\,\frac{f'(x)}{f(x)}$$

$$\text{Relative error in the input} = \frac{\Delta x}{x}$$

$$\text{Condition Number} = CN = \frac{h\, f'(x) / f(x)}{h / x} \approx x\,\frac{f'(x)}{f(x)}$$

Relative error in the functional value can be much larger or smaller than that (relative error in) the input.

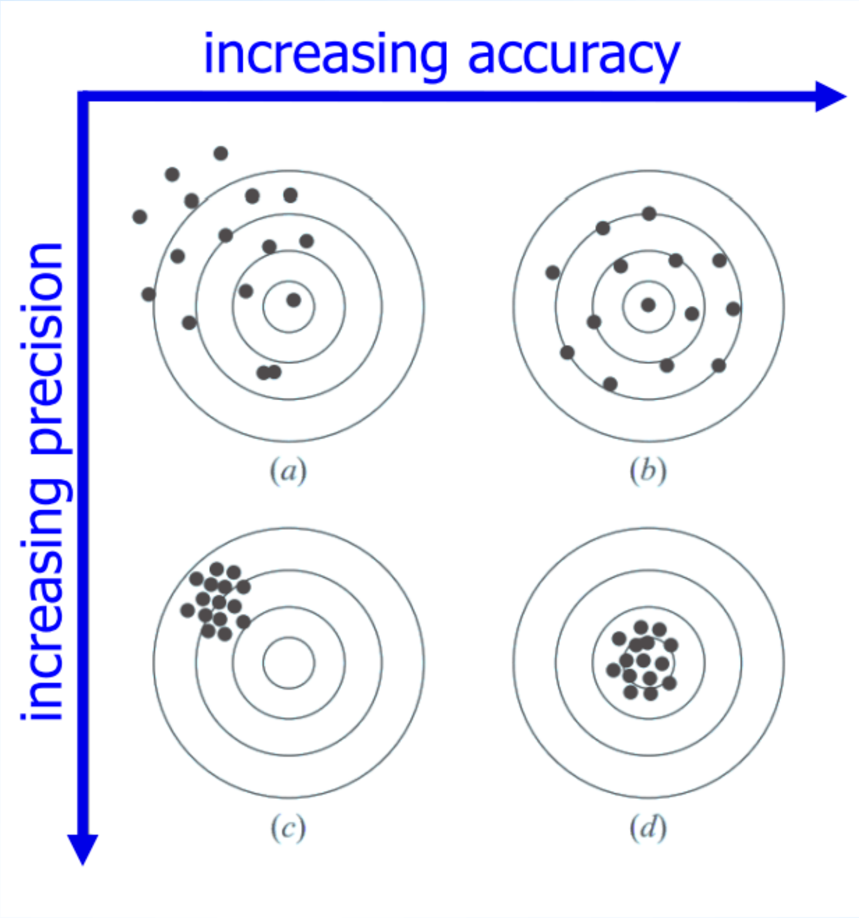
Example 7: Find the condition number, CN, for $f(x) = \dfrac{x}{1 - x}$

(a) When x = 0.93 and

(b) When x = - 0.93

(c) Comment on the results by evaluating f(x) near 0.93 and - 0.93

ACCURACY and PRECISION



ACCURACY

How closely computed or measured values agree with the true value

PRECISION

How closely computed or measured values agree with each other

ERRORS

True Error => usually not known

$$E_{true}(x) = f(x) - P_n(x)$$

Estimated Error => approximated

Approximate Error => estimated

$$E_{est}(x) \cong R$$

True Relative Error => relative to the true value

$$E_{true}(x) = \frac{f(x) - P_n(x)}{f(x)}$$

Estimated Relative Error => relative to the approximate value

$$E_{est}(x) \cong \frac{R}{P_n(x)}$$

ERRORS

Truncation Error => series representation

Round-off Error => lost precision

Deterministic Error => due the measuring instrument

Statistical (stocastic) Error => result of experiments

ERRORS

Programming Errors => **Syntax error** (no compilation)

=> **Run-time error** (compiles, but stops running)

=> **Logical error** (compiles and runs, but the output is wrong)

Read the article «Errors and Error Estimation.pdf» on the Moodle.

Brook Taylor (1685-1731) was educated at St. John's College of Cambridge University, entering in 1701 and graduating in 1709. He published what we know as Taylor's Theorem in 1715, although it appears that he did not entirely appreciate its larger importance and he certainly did not bother with a formal proof. He was elected a member of the prestigious Royal Society of London in 1712.

Taylor acknowledged that his work was based on that of Newton and Kepler and others, but he did not acknowledge that the same result had been discovered by Johann Bernoulli and published in 1694. (But then Taylor discovered integration by parts first, although Bernoulli claimed the credit.)

COLIN MACLAURIN, MATH. PROF. EDIN.

Colin Maclaurin (1698-1746) was born and lived almost his entire life in Scotland. Educated at Glasgow University, he was professor of mathematics at Aberdeen from 1717 to 1725 and then went to Edinburgh. He worked in a number of areas of mathematics, and is credited with writing one of the first textbooks based on Newton's calculus, Treatise of fluxions (1742). The Maclaurin series appears in this book as a special case of Taylor's series.

Jožef *Štefan*
Carinthian Slovene Physicist
1835-1893

Ludwig Boltzmann
Austrian physicist
1844-1906

Jean Baptiste Joseph Fourier

French Mathematician

1768 – 1830