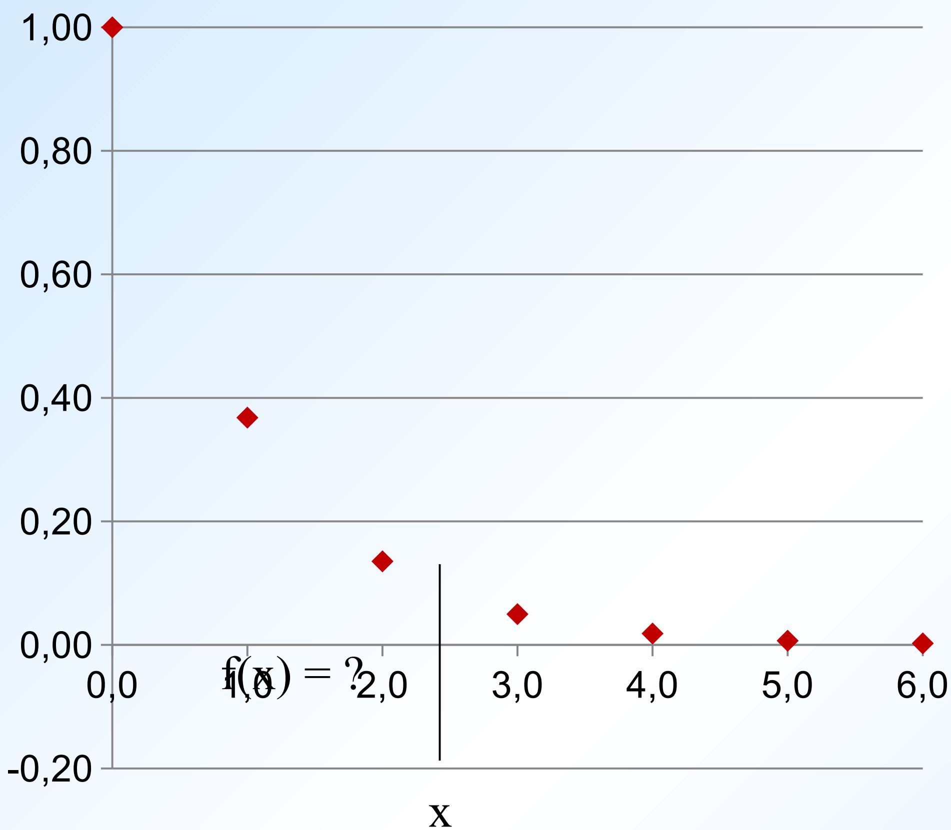
INTERPOLATION

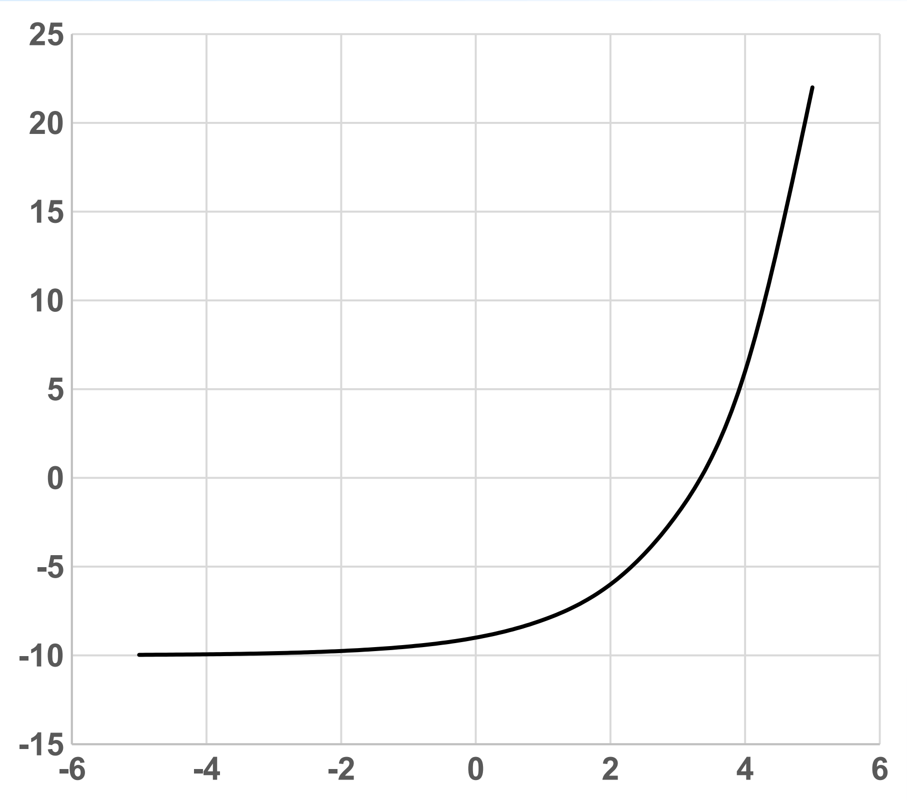


In engineering and science, one often has a number of data points, obtained by sampling or experimentation, which represent the values of a function, f(x), for a limited number of values of the independent variable, x. It is often required to **interpolate**, i.e., estimate the value of that function for an intermediate value of the independent variable.

Remember thermodynamic tables and linear interpolations.

INTERPOLATION

$f(x) = 2^x - 10 \quad \Rightarrow \quad g(x) = ?$



A closely related problem is the approximation of a complicated function by a simple function. Suppose the formula for some given function is known, but too complicated to evaluate efficiently. A few data points from the original function can be interpolated to produce a simpler function which is still fairly close to the original. The resulting gain in simplicity may outweigh the loss from interpolation error.

INTERPOLATING POLYNOMIALS

Given a set of data points (tabulated data) $[x_i , f(x_i)]$, $i = 0$ to N

(or $f(x)$ in (a , b) converted to a set of $N+1$ data points)

No random statistical errors.

Question: Find $f(x)$, x not being one of the data points.

Answer: Replace the data with a polynomial of some chosen, small degree, n,

$$f(x) \cong P_n(x) = c_1 + c_2\, x + c_3\, x^2 + \ldots + c_n\, x^n = g(x)$$

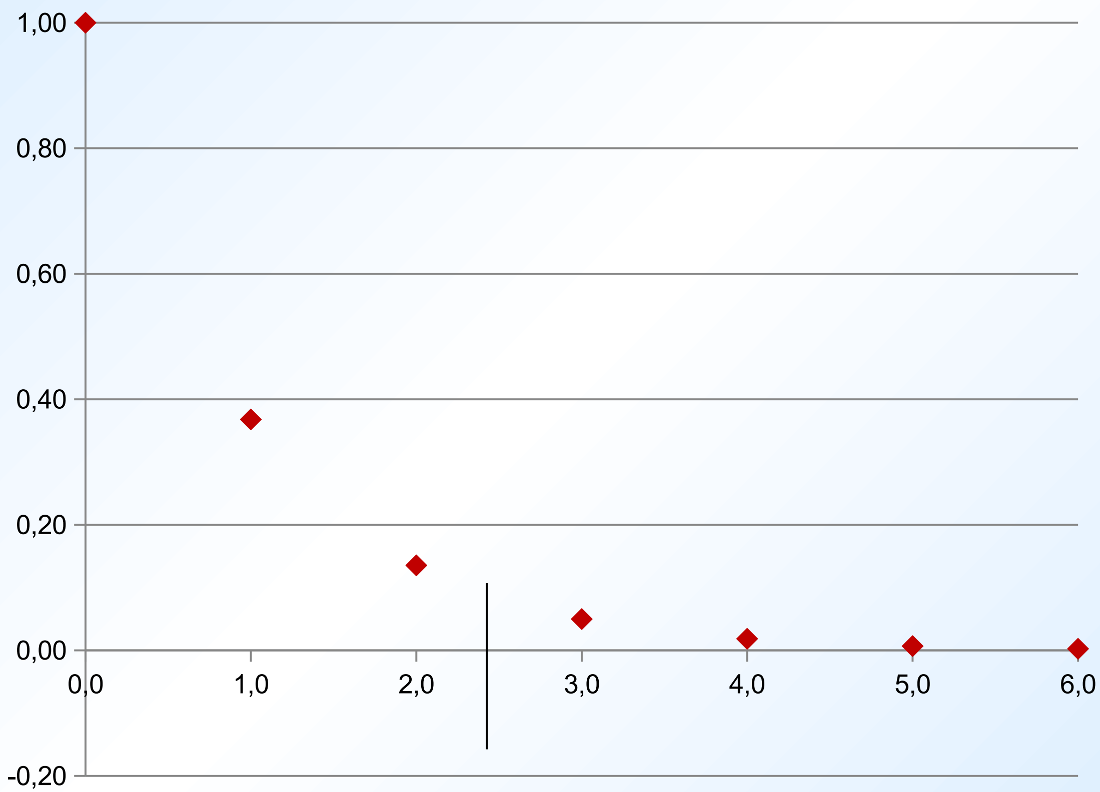
Find all the constants, c_i's, and evaluate $P_n(x)$.

Questions: How do you find the constants, c_i's?

And how do you find or estimate the error?

Example:

x_i	0	1	2	3	4	5	6
$f(x_i)$	1	0.368	0.135	0.050	0.018	0.007	0.002



At $x = 2.5$

$f(x) = f(2.5) = ?$

Error = ?

Solution:

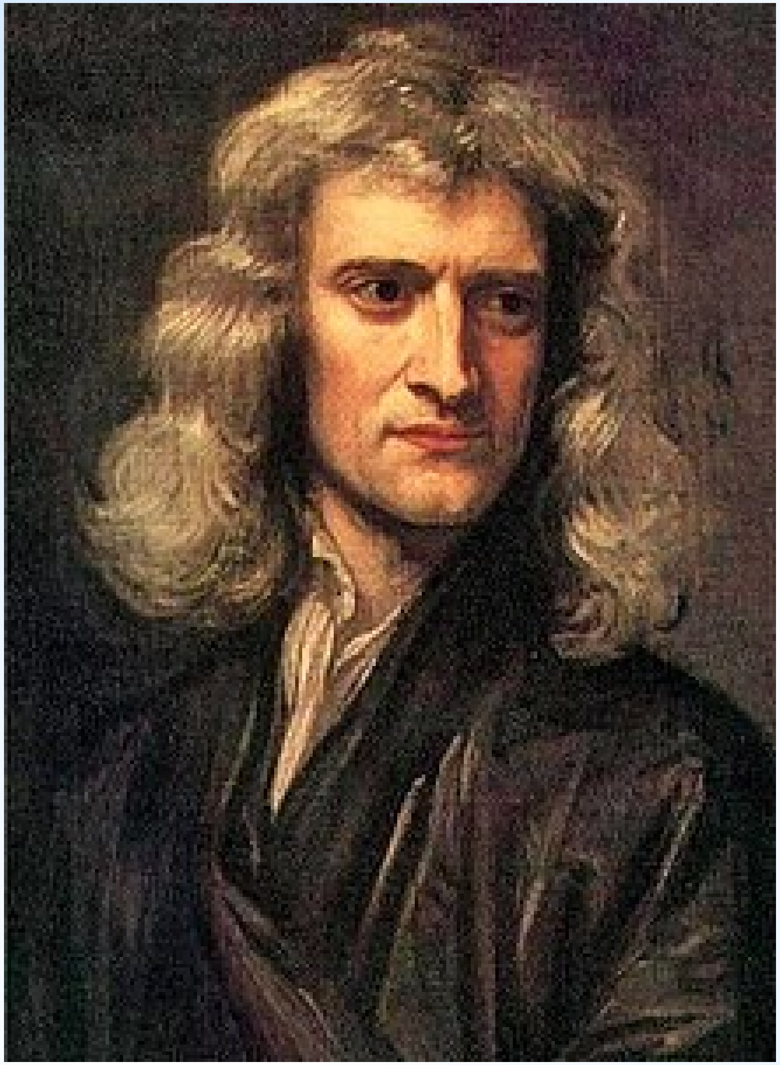
- Choose a polynomial of some (small) degree, $P_n(x)$;
- Decide on the range (the data points to be used);
- Find the constants of the polynomial;
- Evaluate the polynomial at $x = 2.5$. This is the interpolation;
- Estimate the error.

Questions: How do you find the constants, c_i's?

and, how do you find (or estimate) the error?

Answer: Use Newton-Gregory difference tables

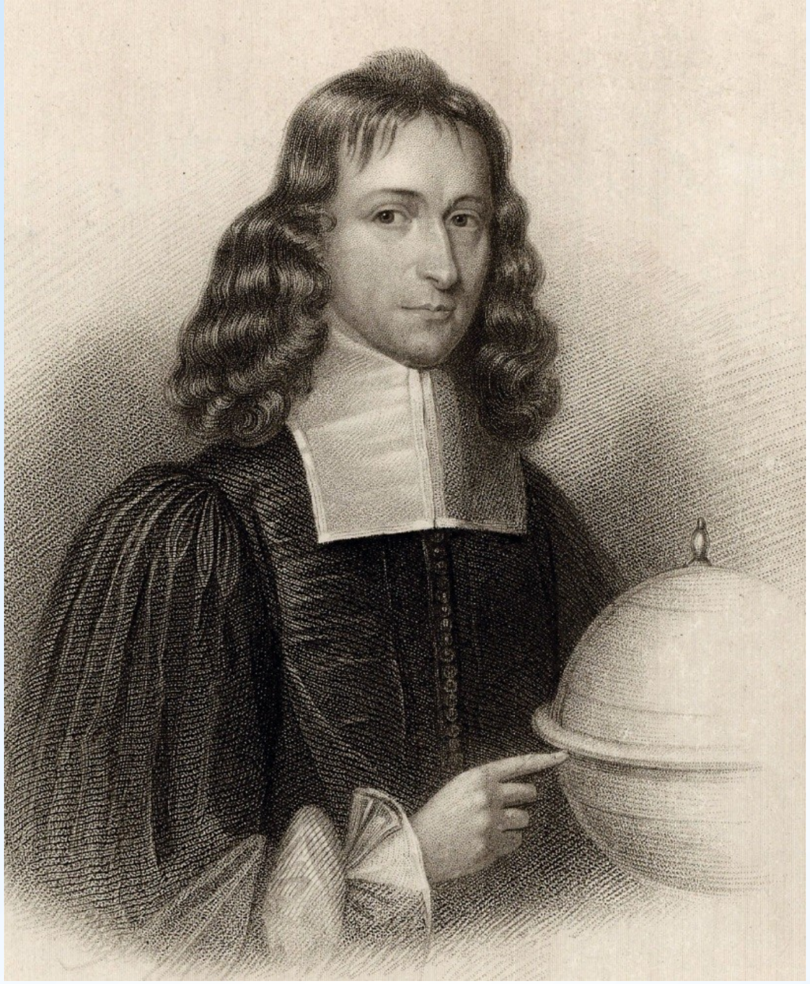
- Forward
- Backward
- Central



Isaac Newton

English Scientist

1642 - 1726



James Gregory (1638-1675) was born just west of Aberdeen, Scotland, where he first went to university. He later (1664-1668) studied in Italy before returning to Britain. He published a work on optics and several works devoted to finding the areas under curves, including the circle. He knew about what we call Taylor series more than 40 years before Taylor published the theorem, and he might have developed the calculus before Newton did if he had been of a less geometric and more analytical frame of mind.

Forward (and backward) differences:

i	x_i	f_i	Δf	$\Delta^2 f$	$\Delta^3 f$	Δ^4
0	x_0	f_0				
			Δf_0			
1	x_1	f_1		$\Delta^2 f_0$		
			Δf_1		$\Delta^3 f_0$	
2	x_2	f_2		$\Delta^2 f_1$		$\Delta^4 f_0$
			Δf_2		$\Delta^3 f_1$	
3	x_3	f_3		$\Delta^2 f_2$		
			Δf_3			
4	x_4	f_4				

i	x_i	f_i	Δf_i	$\Delta^2 f_i$	$\Delta^3 f_i$	$\Delta^4 f_i$
0	1.0	.0	1.0986	-0.5878	0.4135	–0.3244
1	3.0	1.0986	0.5108	-0.1743	0.0891	
2	5.0	1.6094	0.3365	-0.0852		
3	7.0	1.9459	0.2513			
4	9.0	2.1972				

Newton-Gregory Formulae

Define	Forward Differences	$\Delta f_i = f_{i+1} - f_i$	First forward difference at i
		$\Delta^2 f_i = \Delta f_{i+1} - \Delta f_i$	Second forward difference at i
		$\Delta^n f_i = \Delta^{n-1} f_{i+1} - \Delta^{n-1} f_i$	
	Backward Differences	$\nabla f_i = f_i - f_{i-1}$	First backward difference at i
		$\nabla^2 f_i = \nabla f_i - \nabla f_{i-1}$	Second backward difference at i
		$\nabla^n f_i = \nabla^{n-1} f_i - \nabla^{n-1} f_{i-1}$	

Define $s = \dfrac{x - x_0}{\Delta x} = \dfrac{x - x_0}{h}$ where h is constant

Newton-Gregory Forward Polynomial:

$$P_n(x) = f_0 + s\,\Delta f_0 + \frac{s(s-1)}{2!}\Delta^2 f_0 + \dots + \frac{s(s-1)\dots(s-n+1)}{n!}\Delta^n f_0$$

Error:

$$E(x) = \frac{s(s-1)(s-2)\dots(s-n)}{(n+1)!} h^{n+1} f^{(n+1)}(\xi)$$

$$E(x) = \frac{(x-x_0)(x-x_1)\dots(x-x_n)}{(n+1)!} f^{(n+1)}(\xi) \quad , \quad x_0 < \xi < x_n$$

To estimate the error, replace the $(n+1)^{th}$ derivative with $(n+1)^{th}$ divided difference.

Difference Table for $f(x) = e^{-x}$

x	f (x)				
0	1				
1	0.367879441				
2	0.135335283				
3	0.049787068				
4	0.018315639				
5	0.006737947				
6	0.002478752				

Difference Table for f(x) = e^{-x}

x	f (x)	Δ f			
0	1				
		- 0.632120559			
1	0.367879441				
		- 0.232544158			
2	0.135335283				
		- 0.085548215			
3	0.049787068				
		- 0.031471429			
4	0.018315639				
		- 0.011577692			
5	0.006737947				
		- 0.004259195			
6	0.002478752				

Difference Table for f(x) = e^{-x}

x	f (x)	Δ f	Δ^2 f		
0	1				
		- 0.632120559			
1	0.367879441		0.399576401		
		- 0.232544158			
2	0.135335283		0.146995943		
		- 0.085548215			
3	0.049787068		0.054076785		
		- 0.031471429			
4	0.018315639		0.019893738		
		- 0.011577692			
5	0.006737947		0.007318497		
		- 0.004259195			
6	0.002478752				

Difference Table for $f(x) = e^{-x}$

x	f (x)	Δ f	Δ^2 f	Δ^3 f	
0	1				
		- 0.632120559			
1	0.367879441		0.399576401		
		- 0.232544158		- 0.252580458	
2	0.135335283		0.146995943		
		- 0.085548215		- 0.092919158	
3	0.049787068		0.054076785		
		- 0.031471429		- 0.034183048	
4	0.018315639		0.019893738		
		- 0.011577692		- 0.012575241	
5	0.006737947		0.007318497		
		- 0.004259195			
6	0.002478752				

Difference Table for f(x) = e^{-x}

x	f (x)	Δ f	Δ^2 f	Δ^3 f	Δ^4 f
0	1				
		- 0.632120559			
1	0.367879441		0.399576401		
		- 0.232544158		- 0.252580458	
2	0.135335283		0.146995943		0.1596613
		- 0.085548215		- 0.092919158	
3	0.049787068		0.054076785		0.05873611
		- 0.031471429		- 0.034183048	
4	0.018315639		0.019893738		0.021607807
		- 0.011577692		- 0.012575241	
5	0.006737947		0.007318497		
		- 0.004259195			
6	0.002478752				

Example: At $x = 2.5$ $f(x) = f(2.5) = ?$

Choose a polynomial of some degree, $P_n(x)$: Say, $n = 3$

Which four points? Approximately symmetrical around the point of interest

Find the constants of the polynomial: Use the difference table and the formula

Evaluate the polynomial at $x = 2.5$. This is the interpolation.

How do you estimate the error? Use the difference table and the formula

Difference Table for $f(x) = e^{-x}$

x	f (x)	Δ f	Δ^2 f	Δ^3 f	Δ^4 f
0	1				
		- 0.632120559			
1	0.367879441		0.399576401		
		- 0.232544158		- 0.252580458	
2	0.135335283		0.146995943		0.1596613
x = 2.5	?	- 0.085548215		- 0.092919158	
3	0.049787068		0.054076785		0.05873611
		- 0.031471429		- 0.034183048	
4	0.018315639		0.019893738		0.021607807
		- 0.011577692		- 0.012575241	
5	0.006737947		0.007318497		
		- 0.004259195			
6	0.002478752				

Difference Table for f(x) = e^{-x}

x	f (x)	Δ f	Δ^2 f	Δ^3 f	Δ^4 f
0	1				
		- 0.632120559			
x_0 =1	**0.367879441**		0.399576401		
		- 0.232544158		- 0.252580458	
x_1 = 2	0.135335283		**0.146995943**		0.1596613
x = 2.5	?	- 0.085548215		**- 0.092919158**	
x_2 = 3	0.049787068		0.054076785		0.05873611
		- 0.031471429		- 0.034183048	
x_3 = 4	0.018315639		0.019893738		0.021607807
		- 0.011577692		- 0.012575241	
5	0.006737947		0.007318497		
		- 0.004259195			
6	0.002478752				

Define $$s = \frac{x - x_0}{h} = \frac{x - 1}{1} = x - 1$$

Newton-Gregory Forward Polynomial:

$$P_3(s) = f_0 + s\,\Delta f_0 + \frac{s\,(s - 1)}{2\,!}\,\Delta^2 f_0 + \frac{s\,(s - 1)\,(s - 2)}{3\,!}\,\Delta^3 f_0$$

$$P_3(x) = 0.368 + (x - 1)\,(-\,0.233) + \frac{(x - 1)\,(x - 2)}{2}\,(0.147) + \frac{(x - 1)\,(x - 2)\,(x - 3)}{6}\,(-\,0.093)$$

Error:

$$E\,(x) = \frac{(x - 1)\,(x - 2)\,(x - 3)\,(x - 4)}{4!}\,f^{(4)}(\xi)$$

$$E\,(x) \cong \frac{(x - 1)\,(x - 2)\,(x - 3)\,(x - 4)}{4!}\,\frac{\Delta^4 f_0}{h^4}$$

Difference Table for f (x) = x^2

x	f (x)	Δ f	Δ^2 f	Δ^3 f
0	0			
		1		
1	1		2	
		3		0
2	4		2	
		5		0
3	9		2	
		7		0
4	16		2	
		9		
5	25			

Difference Table for f (x) = x^2

x	f (x)	Δ f	Δ^2 f	Δ^3 f
0	0			
		1		
1	1		**1.9**	
		2.9		**0.3**
2	**3.9**		**2.2**	
		5.1		**- 0.3**
3	9		**1.9**	
		7		**0.1**
4	16		2	
		9		
5	25			

Polynomial Interpolation with Unequally-spaced Data

Lagrange Form of a Polynomial:

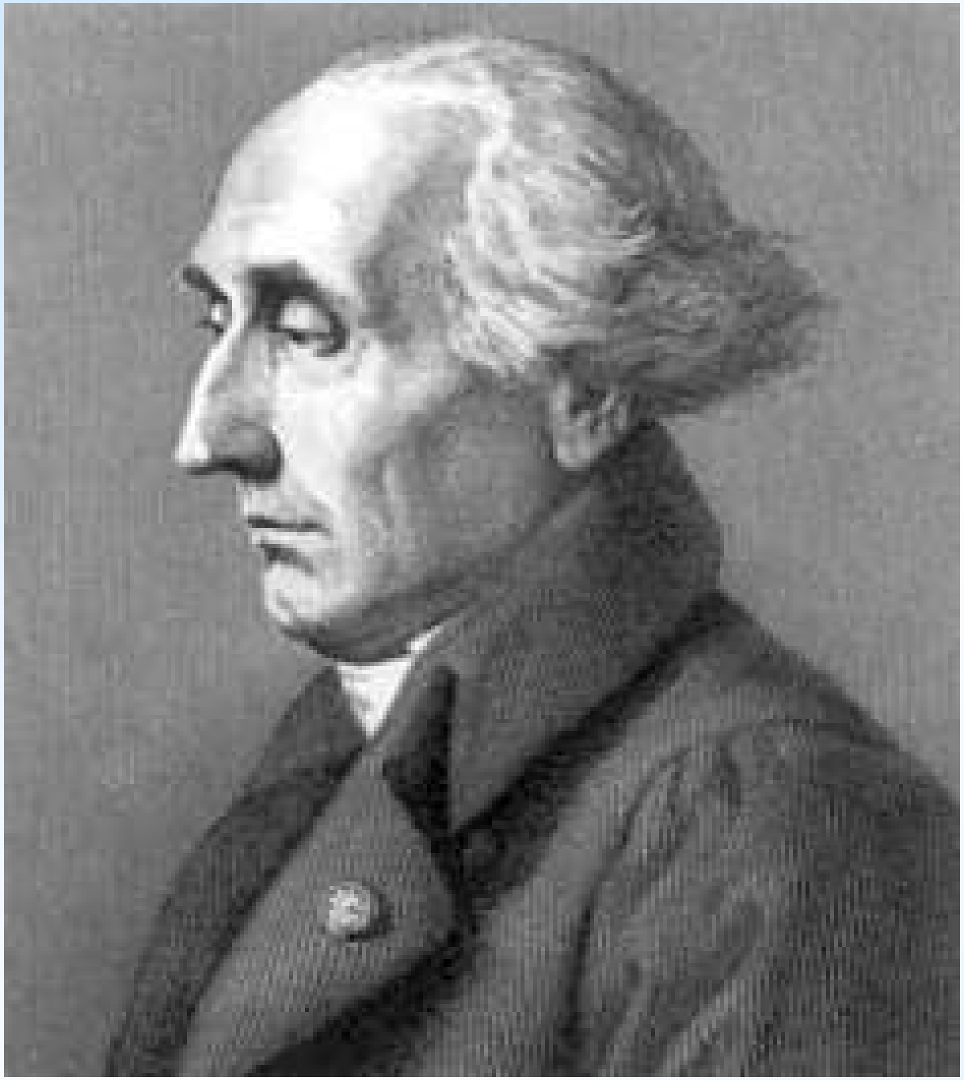
$$P_n(x) = \frac{(x - x_1)(x - x_2)\ldots(x - x_{n-1})(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_{n-1})(x_0 - x_n)} f(x_0)$$

$$+ \frac{(x - x_0)(x - x_2)\ldots(x - x_{n-1})(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_{n-1})(x_1 - x_n)} f(x_1)$$

$$+ \ldots$$

$$+ \frac{(x - x_0)(x - x_1)\ldots(x - x_{n-2})(x - x_n)}{(x_{n-1} - x_0)(x_{n-1} - x_1)\ldots(x_{n-1} - x_{n-2})(x_{n-1} - x_n)} f(x_{n-1})$$

$$+ \frac{(x - x_0)(x - x_1)\ldots(x - x_{n-2})(x - x_{n-1})}{(x_n - x_0)(x_n - x_1)\ldots(x_n - x_{n-2})(x_n - x_{n-1})} f(x_n)$$



Joseph-Louis Lagrange

French Mathematician

1736 - 1813)

Joseph-Louis Lagrange (1736-1813) was born in Turin, Italy, of French parents. At a very young age he was made a professor at the Royal Artillery School in Turin. In 1766, he succeeded his friend and mentor Euler as the Director of the Mathematics Section of the Berlin Academy. In 1787 he accepted a similar position at the Paris Academy of Sciences, where he was active in establishing two very influential French schools, the Ecole Polytechnique, and the École Normale. In his later years, he was granted a number of honors by the French emperor. Napoleon Bonaparte.

His results on interpolation were a major part of his work while in Berlin, but the interpolation formula ascribed to him here was not published until 1795, after he had moved to France. Ironically, Lagrange claims to have gotten the idea from some of Newton's work.

Divided differences:

$$D f_i = \frac{f_{i+1} - f_i}{x_{i+1} - x_i}$$

First-order forward divided difference

$$D^2 f_i = \frac{D f_{i+1} - D f_i}{x_{i+2} - x_i}$$

Second-order forward divided difference

$$D^n f_i = \frac{D^{n-1} f_{i+1} - D^{n-1} f_i}{x_{i+n} - x_i}$$

n'th-order forward divided difference

Divided Difference Table for $f(x) = e^{-x}$

x	f (x)			
0	1			
1	0.367879441			
3	0.049787068			
6	0.002478752			

Divided Difference Table for $f(x) = e^{-x}$

x	f (x)	D f		
0	1			
		-0.632120559		
1	0.367879441			
		-0.159046186		
3	0.049787068			
		-0.015769439		
6	0.002478752			

Divided Difference Table for f(x) = e^{-x}

x	f (x)	D f	D^2 f	
0	1			
		-0.632120559		
1	0.367879441		0.157691457	
		-0.159046186		
3	0.049787068		0.02865535	
		-0.015769439		
6	0.002478752			

Divided Difference Table for $f(x) = e^{-x}$

x	f (x)	D f	D² f	D³ f
0	1			
		-0.632120559		
1	0.367879441		0.157691457	
		-0.159046186		-0.021506018
3	0.049787068		0.02865535	
		-0.015769439		
6	0.002478752			

Newton-Gregory Interpolating Polynomial with Forward Divided Differences:

$$P_n(x) = f(x_i) + Df_i\,(x - x_i) + D^2 f_i\,(x - x_i)(x - x_{i+1}) + \ldots + D^n f_i\,(x - x_i)(x - x_{i+1}) \ldots (x - x_{i+n-1})$$

Error:

$$E(x) = f(x) - P_n(x) = \prod_{i=0}^{n} (x - x_i)\ \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

$$E(x) \cong \prod_{i=0}^{n} (x - x_i)\ D^{n+1} f_i$$

Example: At $x = 2.5$ $f(x) = f(2.5) = ?$

Choose a polynomial of some degree, $P_n(x)$: Say, $n = 3$

Which four points? There are only four points in the given data. So, we use them all.

Find the constants of the polynomial: Use the divided difference table and the formula for divided differences

Evaluate the polynomial at $x = 2.5$. This is the interpolation.

How do you estimate the error? Use the divided difference table and the related formula

Newton-Gregory Forward Polynomial:

$$P_3(x) = f(x_0) + (x - x_0)\, Df_i + (x - x_0)(x - x_1)\, D^2 f_i + (x - x_0)(x - x_1)(x - x_2)\, D^3 f_i$$

$$P_3(x) = 1 + x\,(-0.6321206) + x\,(x - 1)\,(0.15769146) + x\,(x - 1)(x - 3)\,(-0.0215060)$$

Error:

$$E(x) = \frac{(x - 1)(x - 2)(x - 3)(x - 4)}{4!}\, f^{(4)}(\xi)$$

$$E(x) \cong (x - 1)(x - 2)(x - 3)(x - 4)\, D^4 f_0$$

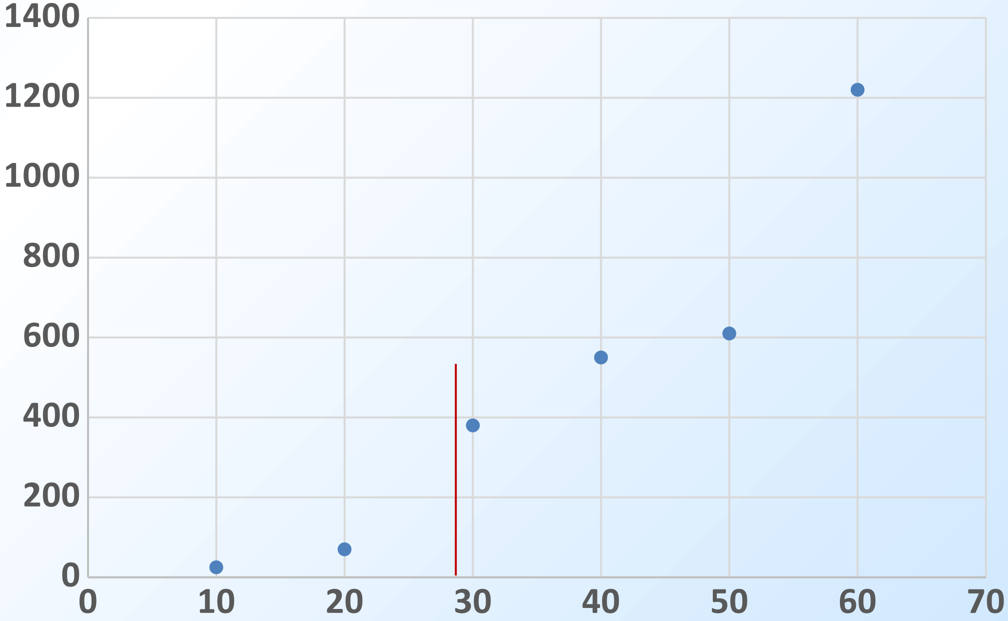
Note that the fourth divided difference, D^4f, does not exists. So, what do you do?

Example: An object is suspended in a wind tunnel and the force is measured for various levels of wind velocity. The results are tabulated below.

V, m/s	10	20	30	40	50	60
F, N	25	70	380	550	610	1220

(a) Find the force at V = 32 m/s using a second-degree interpolating polynomial, $P_2(V)$.

(b) Estimate the error.



Difference table:

V	F	ΔF	Δ^2F	Δ^3F	Δ^4F	Δ^5F
10	25					
		45				
20	70		265			
		310		-405		
30	380		-140		435	
		170		30		195
40	550		-110		630	
		60		660		
50	610		550			
		610				
60	1220					

Interpolating polynomial: $$P_2(V) = 70 + 31\,(V - 20) - 0.7\,(V - 20)\,(V - 30)$$

$$P_2(32) = 425.2 \text{ N}$$

Error estimate: $$E(x) = \frac{(V - V_0)\,(V - V_1)\,(V - V_2)}{3!}\;\frac{\Delta^3 F}{h^3} = -\,0.96$$

Divided difference table:

V	F	DF	D^2F	D^3F	D^4F	D^5F
10	25					
		4.5				
20	70		1.325			
		31		-0.0675		
30	380		-0.7		0.00181	
		17		0.005		0.00001625
40	550		-0.55		0.00263	
		6		0.11		
50	610		2.75			
		61				
60	1220					

Interpolating polynomial: $P_2(V) = 70 + 31\,(V - 20) - 0.7\,(V - 20)\,(V - 30)$

$$P_2(32) = 425.2 \text{ N}$$

Error estimate: $E(x) = (V - V_0)\,(V - V_1)\,(V - V_2)\,(D^3F) = -\,0.96$

Rules of Thumb for Polynomial Interpolation

- The error of polynomial interpolation is reduced by making the range as symmetric as possible around the point of interpolation.
- The truncation error will decrease but round-off will increase as the degree of the polynomial increases.
- A local irregularity in the tabulated data can cause amplified distortions at points far from the irregularities. Such amplification increases with the degree of the interpolating polynomial.
- If there is a choice, smaller step size, h, leads to higher accuracy.