Numerical Integration

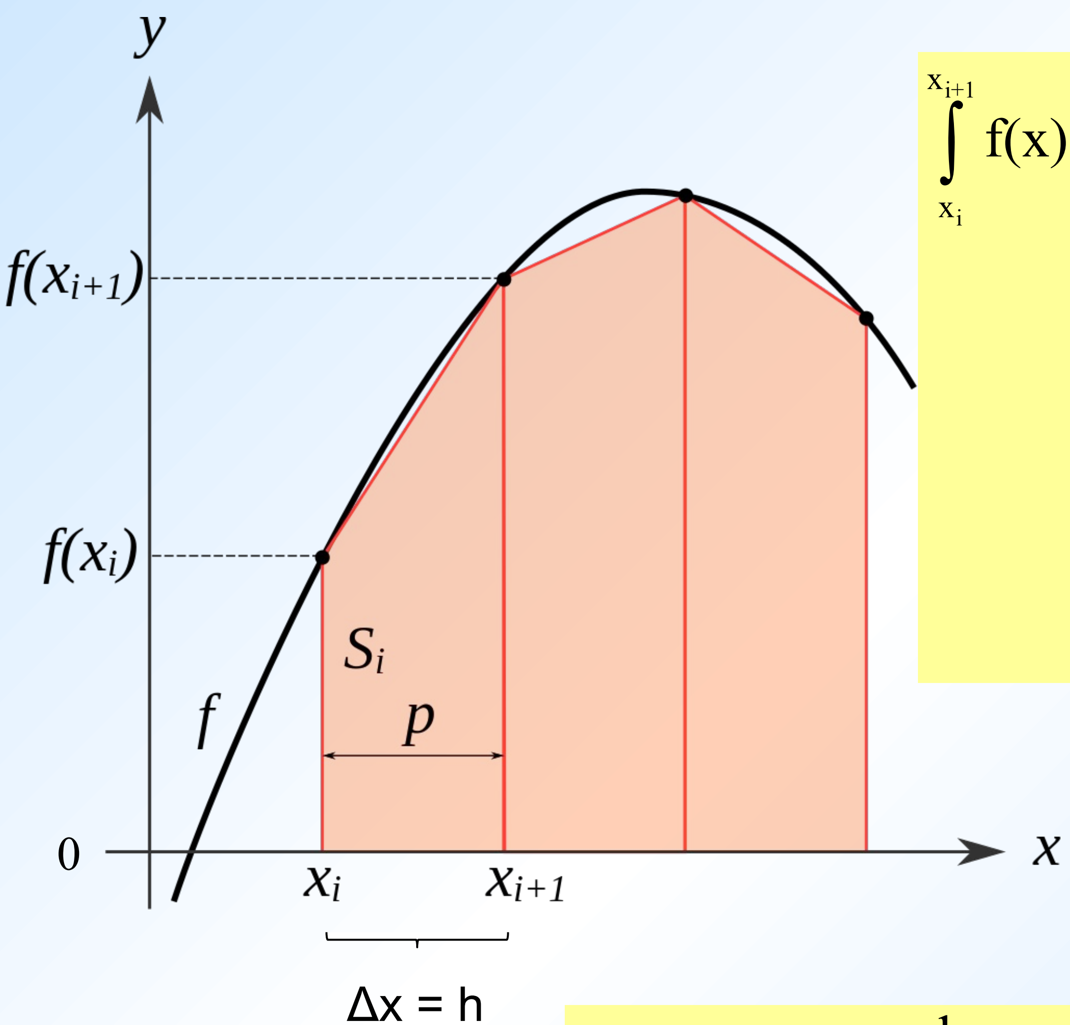
Given a complicated f(x) or a set of tabulated values, x_i's and corresponding $f(x_i)$'s:

Question: For a given range, [a,b]

$$\int_a^b f(x)\, dx = ?$$

Answer: Replace f (x) or the tabulated data with a simple function g(x) and operate on g(x), instead

$$f(x) \cong g(x) \quad , \quad \int_a^b f(x)\, dx \cong \int_a^b g(x)\, dx$$



$$\int_{x_i}^{x_{i+1}} f(x)\,dx \cong \int_{x_i}^{x_{i+1}} P_1(x)\,dx$$

$$\cong h\,f(x_i) + \frac{h}{2}\left[f(x_{i+1}) - f(x_i)\right]$$

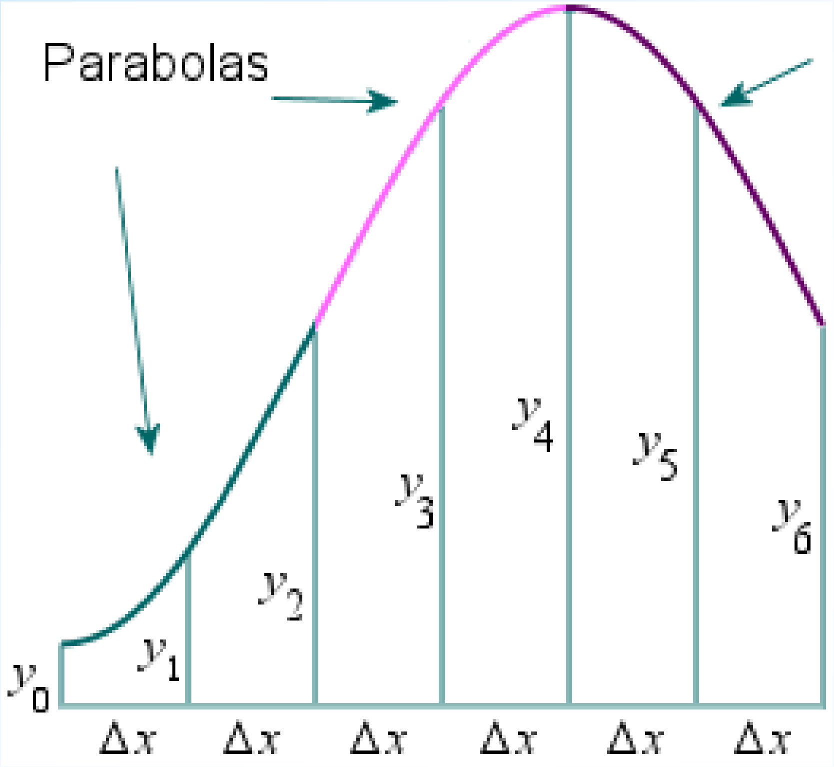
$$\cong \frac{h}{2}\left[f(x_i) + f(x_{i+1})\right]$$

$$\cong \frac{h}{2}\left[f_i + f_{i+1}\right]$$

Trapezoidal Rule

$$\int_{x_0}^{x_n} f(x)\,dx \cong \frac{h}{2}\left(f_0 + 2f_1 + 2f_2 + \ldots + 2f_{n-1} + f_n\right)$$

Second-degree Polynomials, $P_2(x)$



$$\int_{x_0}^{x_2} f(x)\,dx \cong \int_{x_0}^{x_2} P_2(x)\,dx$$

$$P_2(x) = f_0 + s\,\Delta f_0 + \frac{s\,(s-1)}{2\,!}\,\Delta^2 f_0$$

$$s = \frac{x - x_0}{h}$$

Simpson's One-third Rule

$$\int_{x_0}^{x_n} f\,(x)\;d\,x \cong \frac{h}{3}\left(f_0 + 4\,f_1 + 2\,f_2 + \ldots + 2\,f_{n-2} + 4\,f_{n-1} + f_n\right)$$

Newton - Cotes Formulae

Trapezoidal Rule (set of first-degree polynomials)

$$\int_{x_0}^{x_n} f(x)\; dx \cong \frac{h}{2}\left(f_0 + 2\,f_1 + 2\,f_2 + \ldots + 2\,f_{n-1} + f_n\right)$$

$$E_{trap} = \frac{-(x_n - x_0)^3}{12\,n^2}\; f''(\xi)$$

Simpson's One-third Rule (set of second-degree polynomials)

$$\int_{x_0}^{x_n} f(x)\; dx \cong \frac{h}{3}\left(f_0 + 4\,f_1 + 2\,f_2 + \ldots + 2\,f_{n-2} + 4\,f_{n-1} + f_n\right)$$

$$E_{1/3} = \frac{-(x_n - x_0)^5}{180\,n^4}\; f^{(4)}(\xi)$$

Simpson's Three-eights Rule (set of third-degree polynomials)

$$\int_{x_0}^{x_n} f(x)\; dx \cong \frac{3\,h}{8}\left(f_0 + 3\,f_1 + 3\,f_2 + 2\,f_3 + \ldots + 2\,f_{n-3} + 3\,f_{n-2} + 3\,f_{n-1} + f_n\right)$$

Thomas Simpson, FRS

British Mathematician

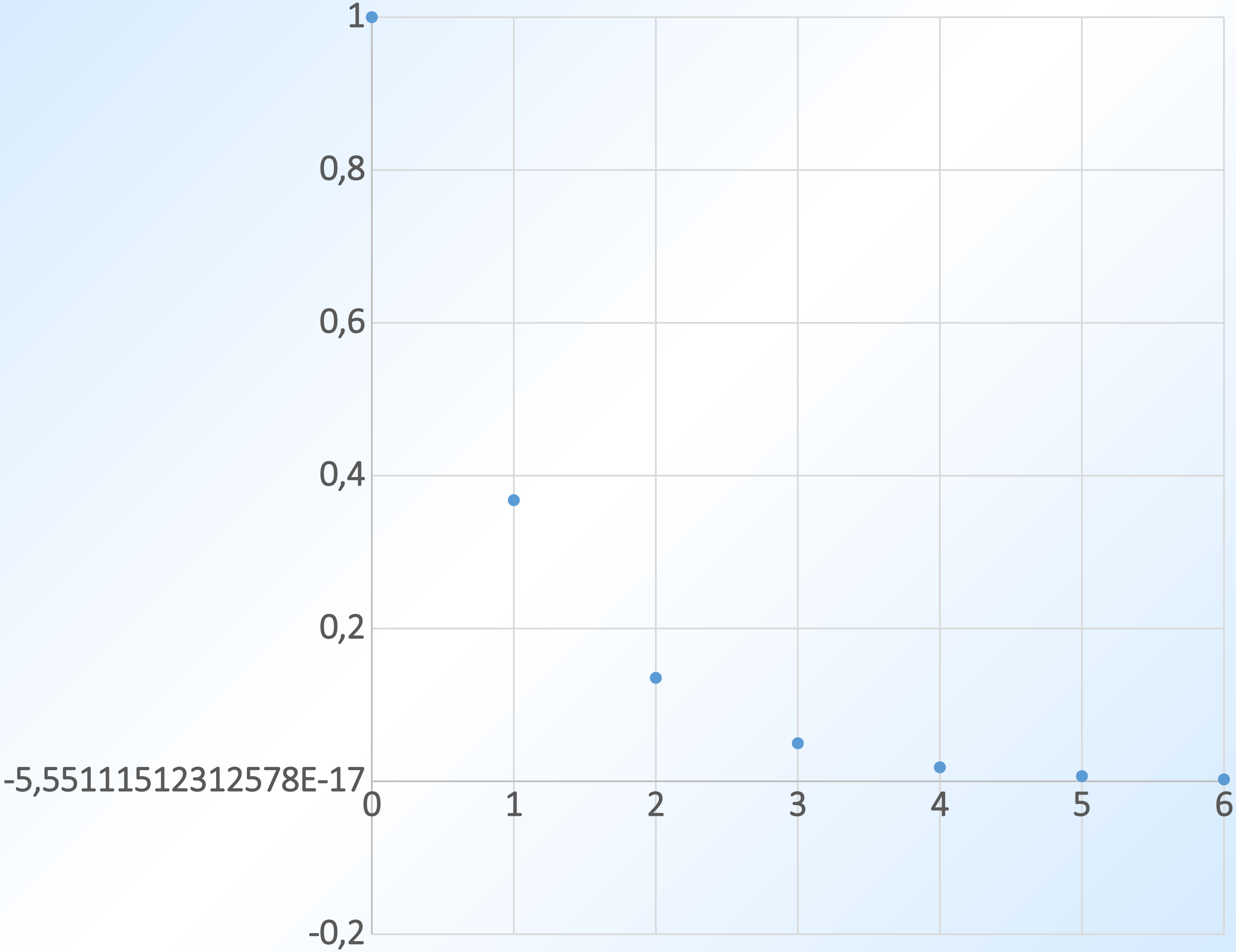
1710 - 1761

Thomas Simpson (1710-1761) was born and died in Leicestershire, England. Little is known of his youth and education, and the legend persists to this day that he kept "low company" with whom he would "guzzle porter and gin." He published an early text on Newton's calculus {The Doctrine and Application of Fluxions) and also worked in probability theory. Most of his mathematics teaching was done privately.

The quadrature rule that bears his name was originally derived by the Italian Bonaventura Cavalieri in 1639, and was known to James Gregory in the late seventeenth century and to Roger Cotes in the early eighteenth century. It was rediscovered by Simpson and published in 1743 in his paper «Mathematical Dissertations on a Variety of physical and analytical subjects».

Difference Table for $f(x) = e^{-x}$

x	f (x)	Δ f	Δ^2 f	Δ^3 f	Δ^4 f
0	1				
		- 0.632120559			
1	0.367879441		0.399576401		
		- 0.232544158		- 0.252580458	
2	0.135335283		0.146995943		0.1596613
		- 0.085548215		- 0.092919158	
3	0.049787068		0.054076785		0.05873611
		- 0.031471429		- 0.034183048	
4	0.018315639		0.019893738		0.021607807
		- 0.011577692		- 0.012575241	
5	0.006737947		0.007318497		
		- 0.004259195			
6	0.002478752				


1
0,8
0,6
0,4
0,2
-5,55111512312578E-17
-0,2
0
1
2
3
4
5
6

Example: Find the integral $I = \int_0^6 e^{-x}\,dx = \int_0^6 f(x)\,dx$ using forward-difference table

	I	True Relative Error
Exact	0.997521	
Trap Rule	1.07929	8 %
Simpson's 1/3 Rule	1.00247	0.5 %
Simpson's 3/8 Rule	1.00757	1 %

Note that Simpson's 1/3 rule requires odd number of data points (even number of panels), multiples of three, and that Simpson's 3/8 rule requires even number of data points (odd number of panels), multiples of four. Otherwise, use a combination of the rules.

Richardson's Extrapolation

Trap-rule formula:

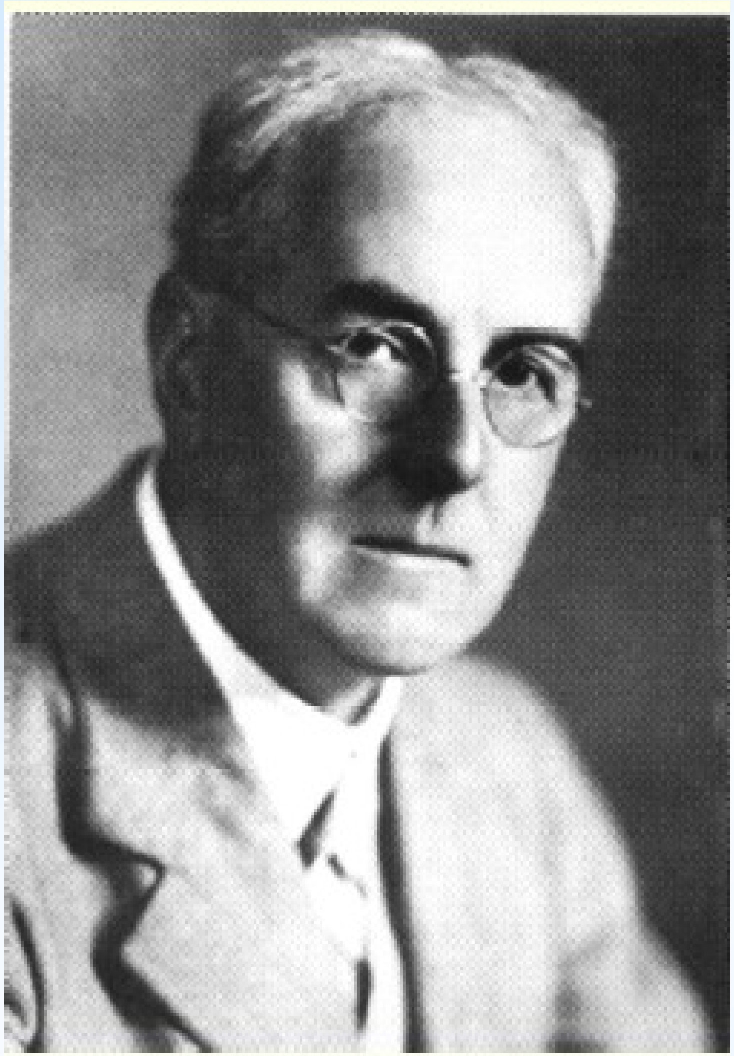
$$I = \int_a^b f(x)\,dx \cong \frac{(b - a)}{n}\left[\frac{1}{2} f_a + \frac{1}{2} f_b + \sum_{i=1}^{n-1} f\left(a + \frac{b - a}{n}\, i\right)\right] - \frac{(b - a)^3}{12\, n^2} f''(\xi)$$

Note that the error is roughly proportional to n^2, where n is the number of panels.

Let's denote the true integral by I*. Find I with trap rule using first with n_1 number of panels and then with n_2 number of panels

$$I^* = I_{n_1} + E_{n_1} = I_{n_1} - \frac{(b - a)^3}{12\, n_1^2} f''(\xi_1)$$

$$I^* = I_{n_2} + E_{n_2} = I_{n_2} - \frac{(b - a)^3}{12\, n_2^2} f''(\xi_2)$$



Lewis Fry Richardson

Britich Scientist

1881 - 1953

Lewis Fry Richardson (1881-1953) was born in Newcastle upon Tyne in Great Britain, and attended several different schools before finishing his education at King's College, Cambridge, in 1903. His professional career spanned a number of different posts in industry, academia, and government science laboratories. He was the first person to suggest using mathematical techniques to predict the weather, by solving the fluid equations that would govern temperature, air pressure, etc. He first did this during World War I, while serving as an ambulance driver in France, long before the development of modem high-speed computers, and it was because of this that he developed the notion of extrapolation methods for the accurate numerical approximation of solutions based on cruder approximations.

$$I^* - I_{n_2} = E_{n_2}$$
$$I^* - I_{n_1} = E_{n_1}$$

=> $$\frac{I^* - I_{n_2}}{I^* - I_{n_1}} = \frac{E_{n_2}}{E_{n_1}}$$ => $$I^* - I_{n_2} = \left(I^* - I_{n_1}\right)\frac{E_{n_2}}{E_{n_1}}$$

Take the ratio of two errors.

$$\frac{E_{n_2}}{E_{n_1}} = \frac{-\dfrac{(b-a)^3}{12\,n_2^2}\,f''(\xi_2)}{-\dfrac{(b-a)^3}{12\,n_1^2}\,f''(\xi_1)} = \left(\frac{n_1}{n_2}\right)^2 \frac{f''(\xi_2)}{f''(\xi_1)} \cong \left(\frac{n_1}{n_2}\right)^2$$

$$I^* - I_{n_2} \cong \left(I^* - I_{n_1}\right)\left(\frac{n_1}{n_2}\right)^2$$

Solve for I*:

$$\boxed{I^* \cong I_{n_1} + \frac{I_{n_2} - I_{n_1}}{1 - \left(\dfrac{n_1}{n_2}\right)^2}}$$

Example: Find the integral $$I = \int_1^3 \left(x^3 - 2x^2 + 7x - 5\right) dx = \int_1^3 f(x)\, dx$$

Exact solution: $I = 20\ \dfrac{2}{3}$

Richardson's extrapolation using $n_1 = 2$ and $n_2 = 4$:

$$I_{n_1} = \frac{3-1}{2}\left[\frac{1}{2} f(1) + \frac{1}{2} f(3) + f(2)\right] = \left[\frac{1}{2} + \frac{25}{2} + 9\right] = 22$$

$$I_{n_1} = \frac{3-1}{4}\left[\frac{1}{2} f(1) + \frac{1}{2} f(3) + f(1.5) + f(2) + f(2.5)\right] = 21$$

$$I^* \cong 22 + \frac{21-22}{1 - \left(\frac{2}{4}\right)^2} = 20\ \frac{2}{3}$$

Why did we get the exact answer?

Romberg Integration Table

$$T_{N,1} \cong \frac{1}{2}\left(T_{N-1,1} + \frac{b-a}{2^{N-1}} \sum_{i=1}^{2^N - 1} f\left(a + \frac{b-a}{2^N}\, i\right)\right) \quad , \quad \Delta i = 2$$

$$T_{N,j} = \frac{4^{j-1}\, T_{N+1,j-1} - T_{N,j-1}}{4^{j-1} - 1} \quad , \quad j = 2, 3, \ldots$$

where $T_{N,j}$: Estimate of an integral

$n = 2^N$ panels

Wemer Romberg

German Mathematician

1909 - 2003

Wemer Romberg (1909-2003) was bom in Berlin, and educated at Ludwig-Maximilian-Universität in Munich, where he got his doctorate in 1933. In 1938, he joined the faculty of the University of Oslo; he spent most of the rest of his working life in Norway. In 1949, he joined the Norwegian Institute of Technology in Trondheim as an associate professor of physics. His paper on what we call Romberg integration was published in 1955. It was not until the late 1960s that the method attracted a lot of attention. See the paper by Jacques Dutka, "Richardson extrapolation and Romberg integration," Hisl. Math., vol. 11 (1984), pp. 3-21, for some of the history of this method.

Example:	$I = \int_0^1 e^{-x}\, dx$	n (Panels)	N	j = 1	j = 2	j = 3
	Exact I = 0.63212	1	0	0.6839	0.6323	0.6321
		2	1	0.6452	0.6321	0.6321
		4	2	0.6354	0.6321	0.6321
		8	3	0.6329	0.6321	
		16	4	0.6323		

Accuracy $\left| T - T \right| \leq \varepsilon$

Monte Carlo Integration

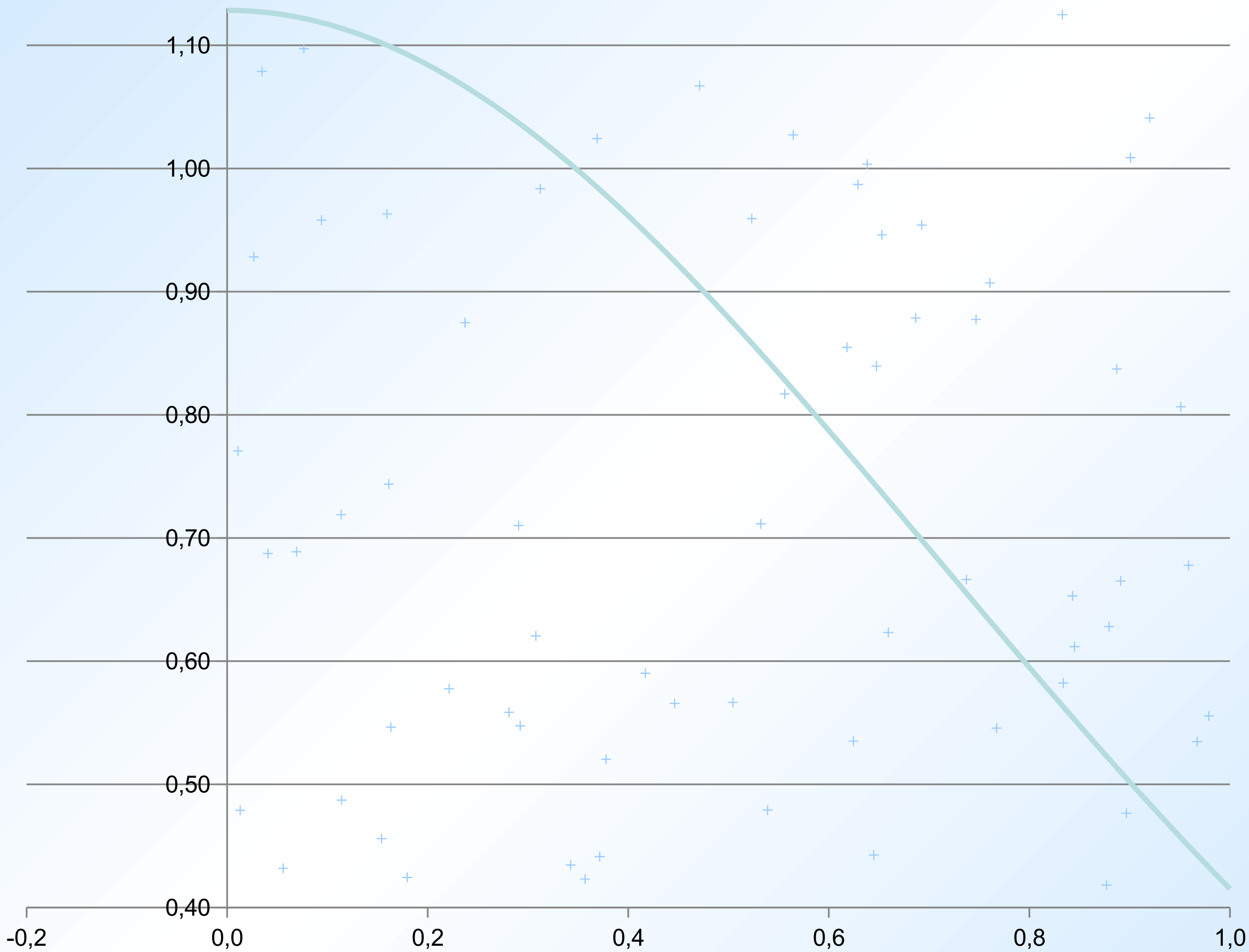
Example:

The error function is defined by

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

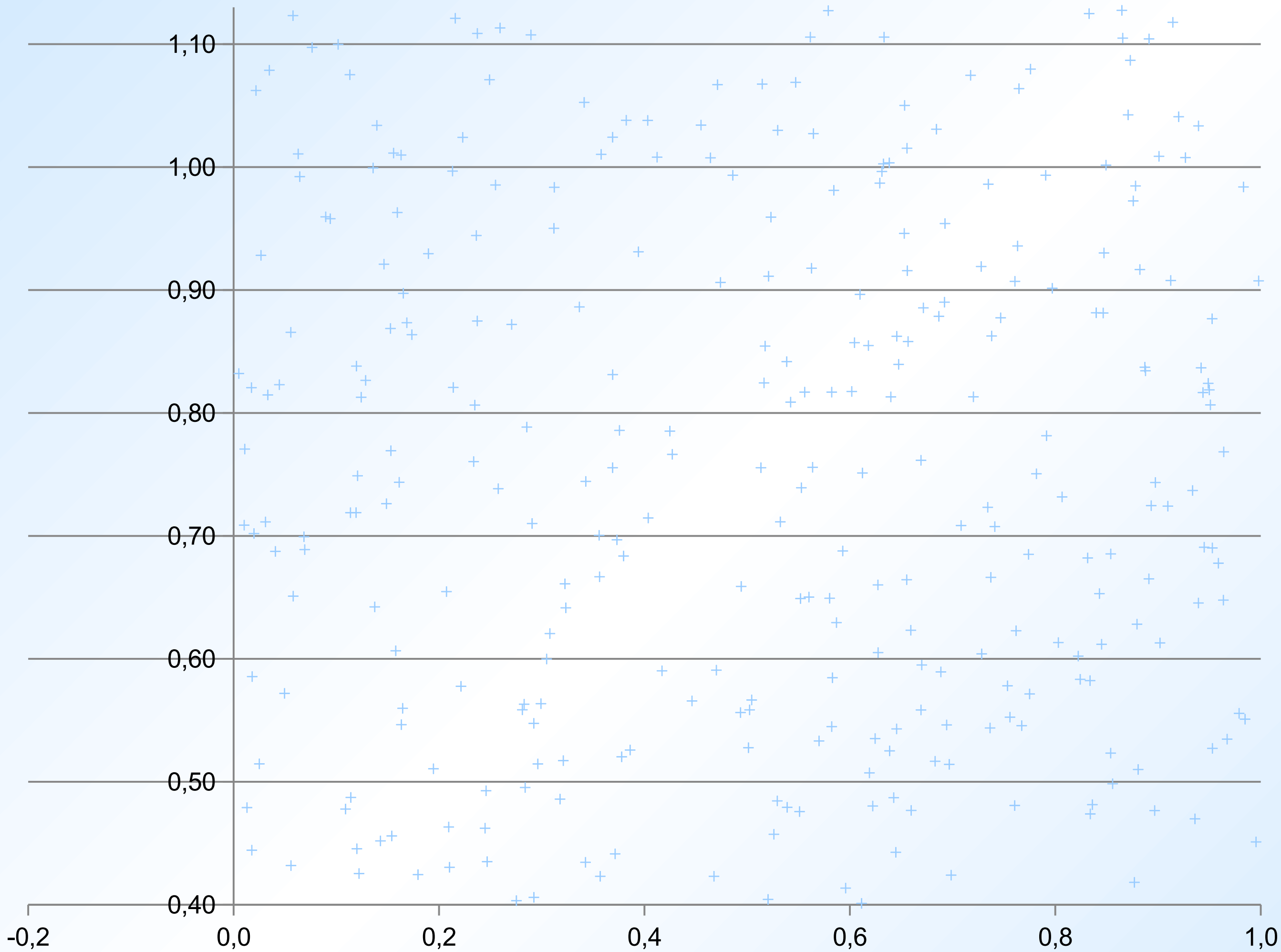
The integral must be obtained, numerically. It is given in tabular form for different values of x, from which Erf(1) = 0.84270

- Use Monte-Carlo method to evaluate the integral. Choose N = 100 and N = 500, and compare with the true value.
- Use Gauss-Legendre quadrature method to evaluate the same integral. Choose N = 3 and N = 6, and compare with the true.



N =100

I = 0.73



N =500

I = 0.81

Gauss - Legendre Quadrature Method

$$I = \int_a^b f(x)\, dx \cong \sum_{i=0}^{n} w_i\, f(x_i)$$

n + 1 values of w_i

n + 1 values of $f(x_i)$

=> There are 2 n + 2 parameters which define a polynomial of degree 2 n + 1

Choose x_i's such that, the sum $\sum_{i=0}^{n} w_i\, f(x_i)$ gives the integral

$\int_a^b f(x)\, dx$ <u>exactly</u> when f (x) is a polynomial of degree 2 n + 1.

Johann Carl Friedrich Gauss

German Mathematician

1777 – 1855

Carl Friedrich Gauss (1777-1855) is widely regarded as one of the greatest mathematicians of all time, in a class with Archimedes, Newton, and Eulcr. A child prodigy who taught himself to read and do arithmetic before beginning elementary school, Gauss attended the Collegium Carolinum Brunswick and the University of Göttingen, graduating in 1800. His doctoral dissertation was accepted in 1801. Briefly supported by his patron, the Duke of Brunswick, Gauss accepted an appointment at Göttingen as director of the observatory in 1804 and as professor of astronomy in 1807.

Like Euler, there are few areas of mathematics that are untouched by Gauss's mind and Gauss's name. Unlike Euler, who published a lot, Gauss published sparingly, often not at all, leaving some of his most significant results in his personal diary. Nonetheless, Gauss has to his credit results in physics, astronomy (including the calculation of the orbit of the planetoid Ceres), number theory, and non-Euclidean geometry. His collected works come to 12 volumes. Gaussian quadrature bears his name because of a paper he presented to the Göttingen Society in 1814, titled «Methodus nova itegralium valores per approximalionem inveniendi».

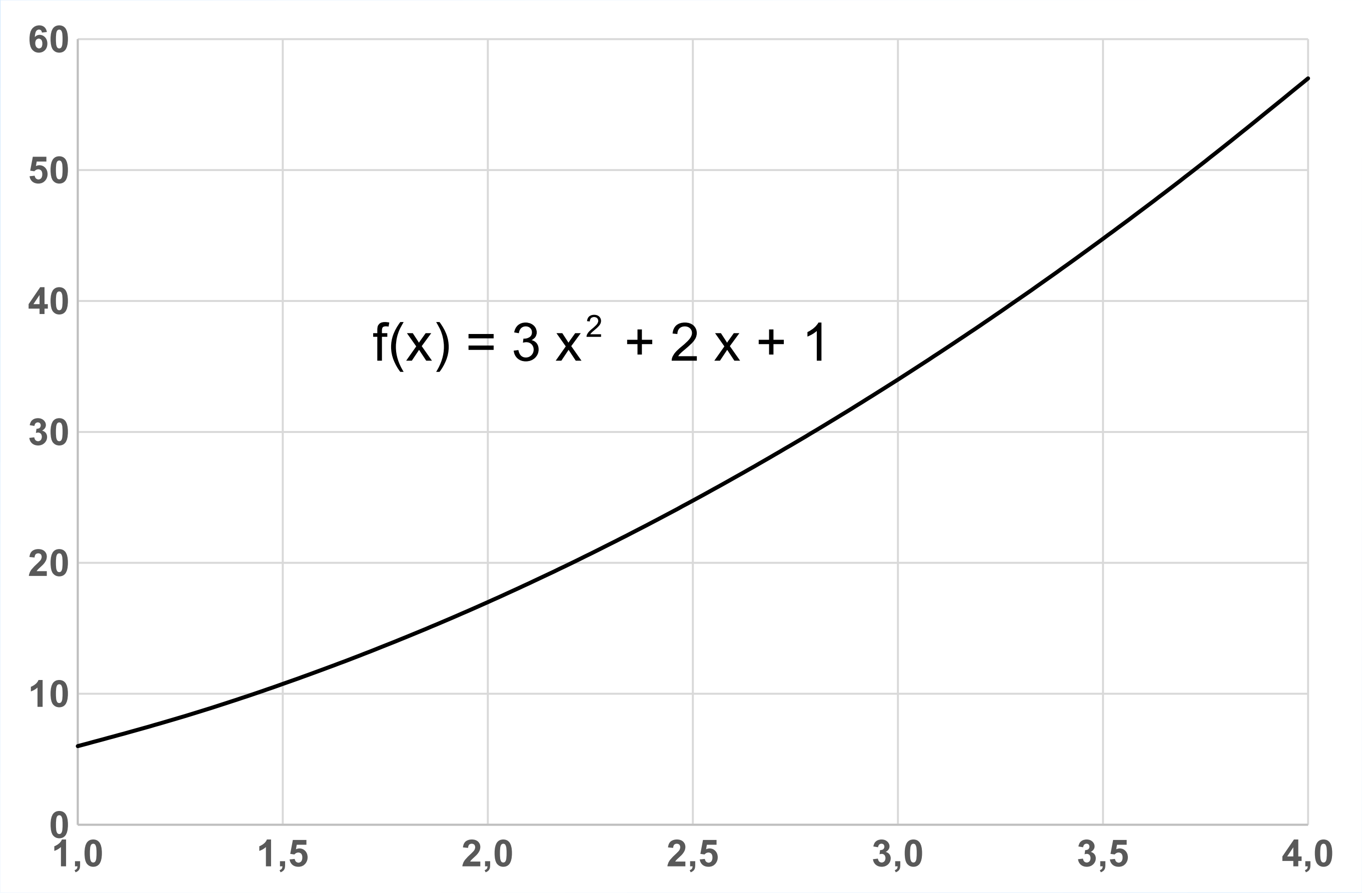
Adrien-Marie Legendre

French Mathematician

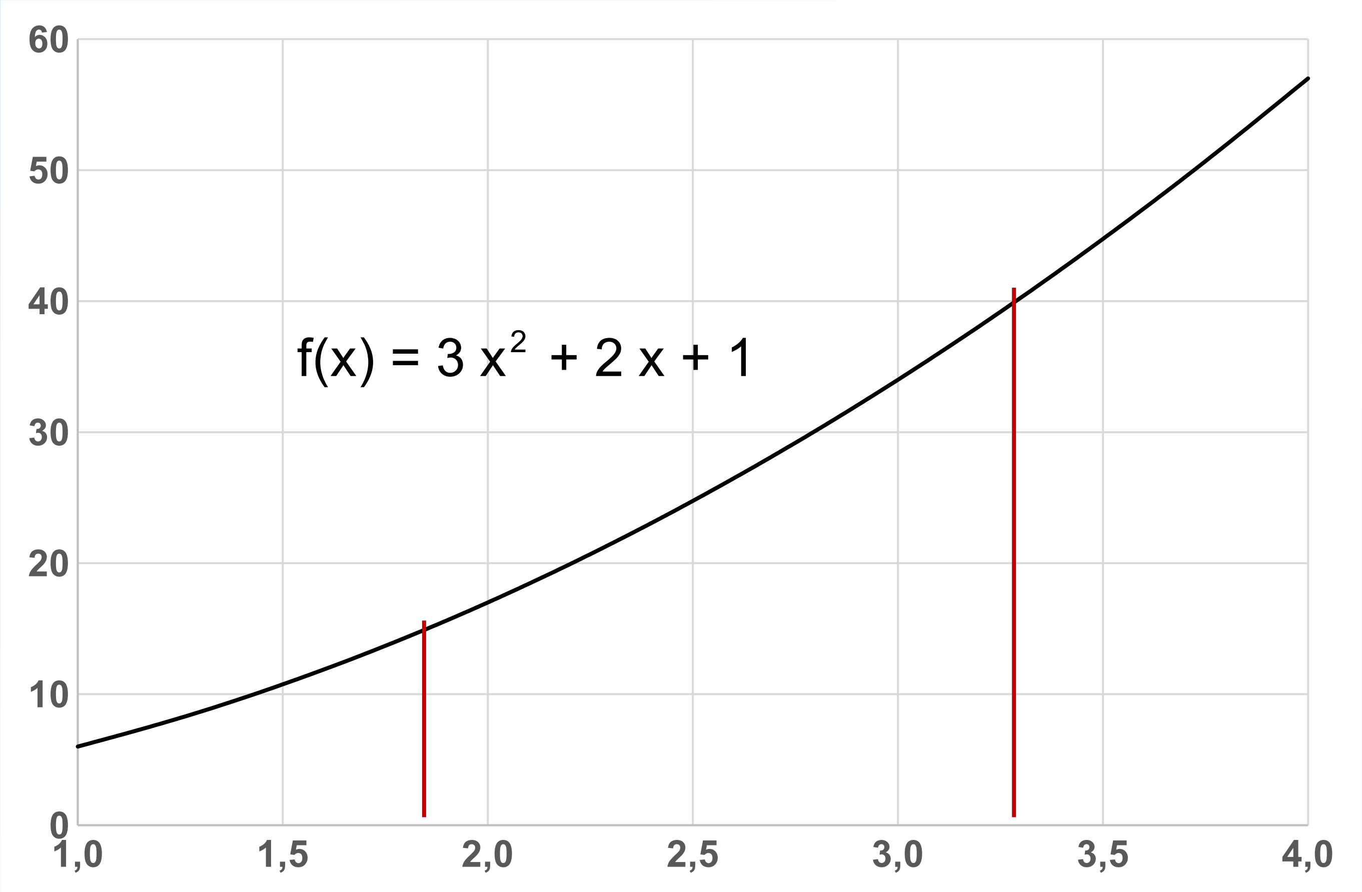
1752 – 1833

Adrien-Marie Legendre (1752-1833) was born and educated in Paris. He contributed greatly to what we now call number theory as well as elliptic function theory. He was the first to publish, in 1805, a description of the method of least squares, although it appears that Gauss had previously worked out much of the same material.

Legendre introduced the polynomials that bear his name in a 1785 paper on the gravitational attraction of spherical bodies. They arise in this context because they are the solutions to an ordinary differential equation that occurs as part of the solution process for the equations of motion in a spherical coordinate system.



$$I_{ex} = \int_1^4 f(x)\,dx = 81 \qquad I_{trap} = \int_1^4 f(x)\,dx = 81.375 \qquad \text{with } h = 0.5 \;\Rightarrow\; 6 \text{ panels}$$



$$I_{ex} = \int_1^4 f(x)\,dx = 81 \qquad I = \sum_{i=1}^{N} w_i\, F(z_i) = I_{ex} \qquad N = 2 \quad w_i = ? \qquad x_i = ?$$

Define a new independent variable: $z = \dfrac{2x - b - a}{b - a}$ $\quad dz = \dfrac{2}{b - a}\, dx$

Find dz and F(z) $\quad I = \int_a^b f(x)\, dx = \int_{-1}^{1} F(z)\, dz$

Choose N

Find nodes (x_i) and weights (w_i) from tabulated data

The nodes (x_i) are the zeros of **Legendre orthogonal polynomials**, which are orthogonal in the range, (-1, +1)

$$I = \int_a^b f(x)\, dx = \int_{-1}^{1} F(z)\, dz \cong \sum_{i=1}^{N} w_i\, F(z_i)$$

Infinite Set of Orthogonal Functions:

Two functions, f(x) and g(x), both real-valued and continous, are called othogonal to each other with respect to a third (weighting) function, w(x), in the range [a,b] if

$$I = \int_a^b w(x)\, f(x)\, g(x)\, dx = 0$$

Examples:

$$\int_0^1 (1) \sin(n\,\pi\,x) \sin(m\,\pi\,x)\, dx = 0 \quad \text{when} \quad n \neq m$$

$$\{\sin(n\pi x)\ ,\ n = 1, 2, \ldots \infty\}$$

$$\int_0^1 (1) \cos(n\,\pi\,x) \cos(m\,\pi\,x)\, dx = 0 \quad \text{when} \quad n \neq m$$

$$\{\cos(n\pi x)\ ,\ n = 0, 1, \ldots \infty\}$$

Legendre polynomials, which form an infinite set, have this property when w(x) = 1 and the range [-1,+1]

$$\int_{-1}^{1} (1)\, P_n(x)\, P_m(x)\, dx = 0 \quad , \quad n \neq m$$

$$P_0 = 1 \qquad P_1 = z \qquad P_2 = \frac{1}{2}(3z^2 - 1) \qquad P_3 = \frac{1}{2}(5z^3 - 3z)$$

$$\{P_n(x)\ ,\ n = 0, 1, \ldots \infty\}$$

ORTHOGONAL POLYNOMIALS

Name	Range of Orthogonality	Weighting Function	Integral Property
Legendre	[- 1 , + 1]	$w(x) = 1$	$\int_{-1}^{+1} w\, P_k\, P_k\, dx = \frac{2}{2k+1}$
Chebyshev	[- 1 , + 1]	$w(x) = \frac{1}{\sqrt{1 - x^2}}$	$\int_{-1}^{+1} w\, T_k\, T_k\, dx = \pi \quad \text{if } k = 0$ $\int_{-1}^{+1} w\, T_k\, T_k\, dx = \frac{\pi}{2} \quad \text{if } k > 0$
Laguerre	[0 , ∞]	$w(x) = x^{\alpha} e^{-x}$	$\int_{0}^{\infty} w\, L_k^{\alpha}\, L_k^{\alpha}\, dx = \frac{\Gamma(\alpha + k + 1)}{k!}$
Hermite	[- ∞ , +∞]	$w(x) = e^{-x^2}$	$\int_{-\infty}^{+\infty} w\, H_k\, H_k\, dx = \sqrt{\pi}\, 2^k\, k!$

Legendre Polynomials

$P_0(x) = 1$

$P_1(x) = x$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

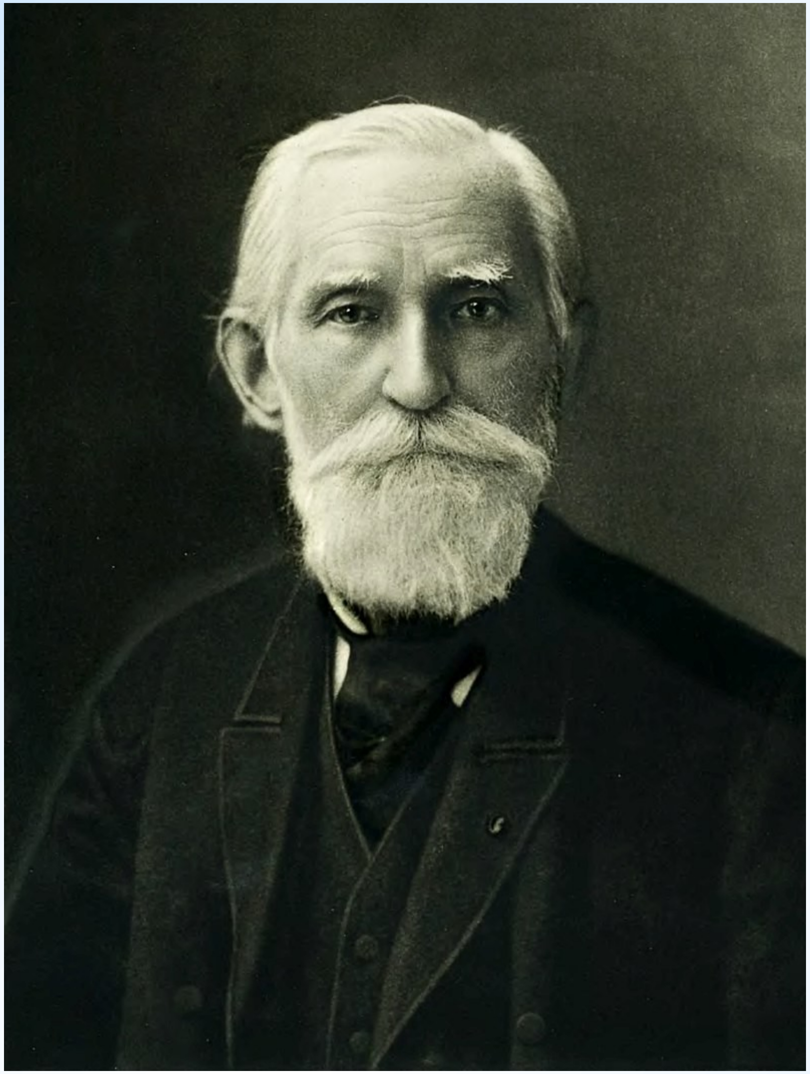
$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

Chebyshev Polynomials

$T_0(x) = 1$	$T_1(x) = x$	$T_2(x) = 2x^2 - 1$
$T_3(x) = 4x^3 - 3x$	$T_4(x) = 8x^4 - 8x^2 + 1$	$T_5(x) = 16x^5 - 20x^3 + 5x$



Pafnuty Lvovich Chebyshev

Rusian Mathematician

1821 - 1894

Pafnuty Lvovich Chebyshev (1821-1894) was born near Borovsk, Russia, southwest of Moscow, and educated at the University of Moscow, from which he was graduated in 1841. From 1847 until his death he lived in St. Petersburg. Many areas of mathematics, from number theory to the theory of equations, were touched by Chebyshev, but he is perhaps best known, at least in applied mathematics, for his work in approximation theory.

Because of the many different ways that the Russian alphabet can be transliterated into the Roman alphabet, there are several different ways to spell Chebyshev's name. The most common alternative is "Tschebyscheff." A marvelously engaging discussion of the perils of transliterating between the Roman and the Russian (Cyrillic) alphabets, as well as a substantial treatment of Chebyshev's life, is contained in The Thread, by Philip J. Davis, a leading mathematician in the area of interpolation and approximation.

Laguerre Polynomials

$$L_0^0(x) = 1$$

$$L_1^0(x) = -x + 1$$

$$L_2^0(x) = \frac{1}{2}(x^2 - 4x + 2)$$

$$L_3^0(x) = \frac{1}{6}(-x^3 + 9x^2 - 18x + 6)$$

$$L_4^0(x) = \frac{1}{24}(x^4 - 16x^3 + 72x^2 - 96x + 24)$$

Hermite Polynomials

$H_0(x) = 1$	$H_1(x) = 2x$	$H_2(x) = 4x^2 - 2$
$H_3(x) = 8x^3 - 12x$	$H_4(x) = 16x^4 - 48x^2 + 12$	$H_5(x) = 32x^5 - 160x^3 + 120x$

Edmond Nicolas Laguerre (1834 - 1886) was born and died in Bar-le-Duc, France. He was educated at the École Polytechnique but did not graduate with a high ranking. He served in the military as an artillery officer for 10 years before returning to the École as a faculty member, where he remained for the rest of his life. Although best known for the orthogonal polynomials that bear his name, and their associated differential equation, Laguerre also published work in analysis, geometry, and abstract linear spaces.

Charles Hermite

French Mathematician

1822 - 1901

Charles Hermite (1822-1901) was bom in Dieuze, France, on Christmas Eve. A poor test-taker, Hermite was not able to pass the exams for his bachelor's degree until the age of nearly 26. He held professional positions at a number of French schools, most notably the École Polytechnique, the École Normale, and the Sorbonne.

Perhaps his best-known mathematical result is the first proof that e is a transcendental number (published in 1873). His name is attached to a number of mathematical ideas and concepts, including the Hermite differential equation, Hermite polynomials, and Hermitian matrices. The idea of interpolating to the derivative as well as to the function values was part of a paper he published in 1878, «Sur la formule d’interpolation de Lagrange», which actually considers interpolation not only of the first derivative, but of higher derivatives as well. Hermite interpolation is sometimes called osculatory interpolation.

Expansion of f(x) in terms of infinite set of orthogonal functions

Suppose the functions, $Q_n(x)$, forms an orthogonal set, n -> inf.

$$f(x) = c_0\, Q_0(x) + c_1\, Q_1(x) + c_2\, Q_2(x) + \ldots + c_n\, Q_n(x) + \ldots.$$

$$\int_a^b w(x)\, Q_n(x)\, f(x)\, dx =$$

$$\int_a^b w(x)\, Q_n(x)\left(c_0\, \cancel{Q_0}(x) + c_1\, \cancel{Q_1}(x) + c_2\, \cancel{Q_2}(x) + \ldots + c_n\, Q_n(x) + \cancel{\ldots.}\right) dx$$

$$= \int_a^b w(x)\, Q_n(x)\left(c_n\, Q_n(x)\right) dx$$

$$c_n = \frac{\int_a^b w(x)\, Q_n(x)\, f(x)\, dx}{\int_a^b w(x)\, Q_n^2(x)\, dx}$$

Define a new independent variable: $z = \dfrac{2\,x - b - a}{b - a}$ $\quad dz = \dfrac{2}{b - a}\,dx$

Find dz and F(z) $\quad I = \int_a^b f(x)\,dx = \int_{-1}^{1} F(z)\,dz$

Choose N

Find nodes (x_i) and weights (w_i) from tabulated data

The nodes (x_i) are the zeros of **Legendre orthogonal polynomials**, which are orthogonal in the range, (-1, +1)

$$I = \int_a^b f(x)\,dx = \int_{-1}^{1} F(z)\,dz \cong \sum_{i=1}^{N} w_i\, F(z_i)$$

Gauss-Legendre Nodes and Weights

N	$\pm z_i$	w_i
2	0.5773502692	1.0000000000
3	0.0000000000	0.8888888889
	0.7745966692	0.5555555556
4	0.3399810436	0.6521451549
	0.8611363116	0.3478548451
5	0.0000000000	0.5688888889
	0.5384693101	0.4786286705
	0.9061798459	0.2369268850
6	0.2386191861	0.4679139346
	0.6612093865	0.3607615730
	0.9324695142	0.1713244924
7	0.0000000000	0.4179591837
	0.4058451514	0.3818300505
	0.7415311856	0.2797053915
	0.9491079123	0.1294849662

N	$\pm z_i$	w_i
8	0.1834346425	0.3626837834
	0.5255324099	0.3137066459
	0.7966664774	0.2223810345
	0.9602898565	0.1012285363
9	0.0000000000	0.3302393550
	0.3242534234	0.3123470770
	0.6133714327	0.2606106964
	0.8360311073	0.1806481607
	0.9681602395	0.0812743884
10	0.1488743390	0.2955242247
	0.4333953941	0.2692667193
	0.6794095683	0.2190863625
	0.8650633667	0.1495513492
	0.9739065285	0.0666713443

Examples:

Integral	Exact Solution	G-L nodes & weights	G-L Solution	Comments
	4	2	4	Exact result
	81	2	81	Exact result
	0.9975212	2	0.8706496	Low accuracy
$I = \int_{0}^{6} e^{-x}\, dx$	0.9975212	4	0.9971697	More accurate

Integral	Exact Solution	G-L nodes & weights	G-L Solution	Comments
$I = \int_{-1}^{1} (3x^2 + 2x + 1)\, dx$	4	2	4	Exact result
$I = \int_{1}^{4} (3x^2 + 2x + 1)\, dx$	81	2	81	Exact result
$I = \int_{0}^{6} e^{-x}\, dx$	0.9975212	2	0.8705496	Low accuracy
$I = \int_{0}^{6} e^{-x}\, dx$	0.9975212	4	0.98860	More accurate

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Example

$$I = \int_0^6 e^{-x}\, dx = \int_0^6 f(x)\, dx \cong \sum_{i=1}^{N=2} w_i\, F(z_i) = w_1\, F(z_1) + w_2\, F(z_2)$$

$$z = \frac{2x - b - a}{b - a} = \frac{2x - 6}{6} = \frac{x - 3}{3} \quad => \quad x = 3(z + 1) \quad dx = 3\, dz$$

$$I = \int_0^6 e^{-x}\, dx = 3 \int_{-1}^{1} e^{-3(z+1)}\, dz \cong 3\left[e^{-3(z_1+1)} + e^{-3(z_2+1)}\right]$$

$$z_{1,2} = \pm\sqrt{\frac{1}{3}} = \pm\, 0.5773502692 \qquad I \cong 0.8706496$$

$$I_{exact} \cong 0.9975212$$

12.7 % error with only 2 nodes and weights

Error expression for Gauss-Legendre quadrature method:

$$E \cong \frac{2^{2N+1}\,[N!]^4}{(2N+1)\,[(2N)!]^3}\, f^{(2N)}(\xi) \qquad , \quad -1<\xi<1$$

where N is the number of nodes and weights used.

GAUSS QUADRATURE METHODS

Name	Quadrature Formula
Gauss-Legendre	$\int_{-1}^{+1} f(x)\, dx \cong \sum_{i=1}^{N} w_i\, f(x_i)$
Gauss-Chebyshev	$\int_{-1}^{+1} \frac{1}{\sqrt{1-x^2}} f(x)\, dx \cong \sum_{i=1}^{N} w_i\, f(x_i)$
Gauss-Laguerre	$\int_{0}^{\infty} e^{-x} f(x)\, dx \cong \sum_{i=1}^{N} w_i\, f(x_i)$
Gauss-Hermite	$\int_{-\infty}^{\infty} e^{-x^2} f(x)\, dx \cong \sum_{i=1}^{N} w_i\, f(x_i)$

GAUSS QUADRATURE METHODS

Name

Nodes and Weights

Gauss-Legendre

N	$\pm x_i$	w_i
2	0.577350269	1.00000000
3	0 0.774596669	0.888888889 0.555555556
4	0.339981043 0.861136312	0.652145155 0.347854845

Gauss-Chebyshev

$$x_i = \cos\left(\frac{i - \frac{1}{2}}{N}\pi\right) \qquad i = 1, 2, \ldots N$$

$$w_i = \frac{\pi}{N} \quad \text{for all } i$$

Gauss-Laguerre

N	x_i	w_i
2	0.58578 64376 3.41421 35624	0.85355 33906 0.14644 66094
3	0.41577 45568 2.29428 03603 6.28994 50829	0.71109 30099 0.27851 77336 0.01038 92565
4	0.32254 76896 1.74576 11012 4.53662 02969 9.39507 09123	0.60315 41043 0.35741 86924 0.03888 79085 0.00053 92947

Gauss-Hermite

N	$\pm x_i$	w_i
2	0.70710 67811	0.88622 69255
3	0. 1.22474 48714	1.18163 59006 0.29540 89752
4	0.52464 76233 1.65068 01239	0.80491 40900 0.08131 28354
5	0. 0.95857 24646 2.02018 28705	0.94530 87205 0.39361 93232 0.01995 32421