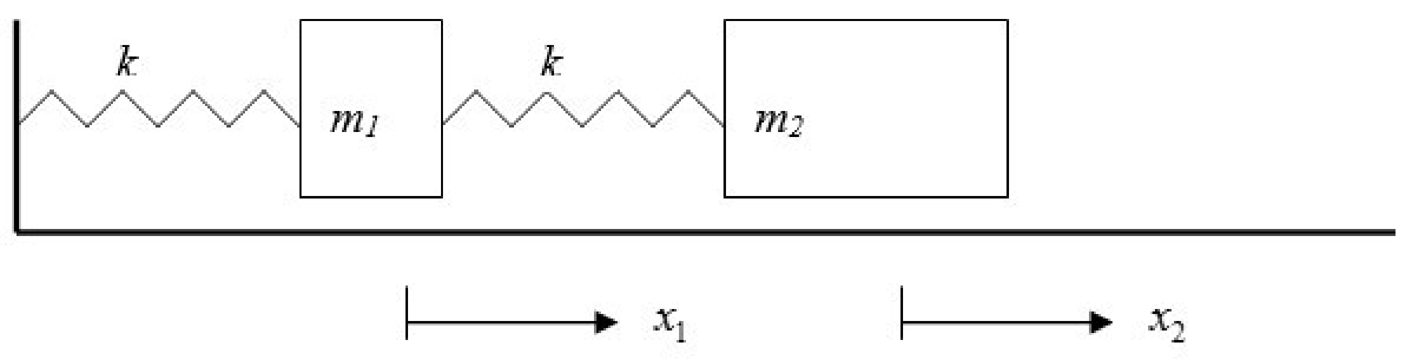
# EIGENVALUES and EIGENVECTORS

The word eigenvalue comes from the German word *Eigenwert* where Eigen means *characteristic* and Wert means *value*.

Consider the spring-mass system:




Assume each of the two mass-displacements to be denoted by $x_1$ and $x_2$, both functions of time, and let us assume each spring has the same spring constant, k.

Apply Newton’s 2nd and 3rd laws of motion to develop a force-balance for each mass:

## Newton’s laws of motion

Newton's **first law** states that every object will remain at rest or in uniform motion in a straight line unless compelled to change its state by the action of an external force. This is normally taken as the definition of **inertia.** The key point here is that if there is **no net force** acting on an object (if all the external forces cancel each other out) then the object will maintain a **constant velocity**. If that velocity is zero, then the object remains at rest. If an external force is applied, the velocity will change because of the force.

## Newton’s laws of motion

The **second law** explains how the velocity of an object changes when it is subjected to an external force. The law defines a **force** to be equal to change in **momentum** (mass times velocity) per change in time. Newton also developed the calculus of mathematics, and the "changes" expressed in the second law are most accurately defined in differential forms. (Calculus can also be used to determine the velocity and location variations experienced by an object subjected to an external force.) For an object with a constant mass **m**, the second law states that the force **F** is the product of an object's mass and its acceleration **a**:

$$F = m\,a$$

For an external applied force, the change in velocity depends on the mass of the object. A force will cause a change in velocity; and likewise, a change in velocity will generate a force. The equation works both ways.

## Newton's laws of motion

The **third law** states that for every action (force) in nature there is an equal and opposite reaction. In other words, if object A exerts a force on object B, then object B also exerts an equal force on object A. Notice that the forces are exerted on different objects. The third law can be used to explain the generation of lift by a wing and the production of thrust by a jet engine.

Apply Newton's 2nd and 3rd laws of motion to develop a force-balance for each mass:

$$m_1 \frac{d^2x_1}{dt^2} = -k\,x_1 + k\,(x_2 - x_1) \qquad m_2 \frac{d^2x_2}{dt^2} = -k\,(x_2 - x_1)$$

These are coupled, second-order, ordinary differential equations to be solved, simultaneously, for $x_1(t)$ and $x_2(t)$.

Re-write the equations:

$$m_1 \frac{d^2x_1}{dt^2} - k\,(-2\,x_1 + x_2) = 0 \qquad m_2 \frac{d^2x_2}{dt^2} - k\,(x_1 - x_2) = 0$$

Let $m_1 = 10$, $m_2 = 20$, and $k = 15$

$$10 \frac{d^2x_1}{dt^2} - 15\,(-2\,x_1 + x_2) = 0 \qquad 20 \frac{d^2x_2}{dt^2} - 15\,(x_1 - x_2) = 0$$

The general solution has the form: $x_i(t) = A_i \sin(\omega t - \emptyset)$

where $A_i$ = amplitude of the vibration of mass i

ω = frequency of vibration

Φ = phase shift

The second derivative becomes: $$\frac{d^2 x_i}{dt^2} = -A_i \omega^2 \sin(\omega t - \emptyset)$$

Substitute into the differential equations:

$$-10 A_1 \omega^2 - 15(-2 A_1 + A_2) = 0 \qquad -20 A_2 \omega^2 - 15(A_1 - A_2) = 0$$

Re-arrange:

$$(-\omega^2 + 3) A_1 - 1.5 A_2 = 0 \qquad -0.75 A_1 + (-\omega^2 + 0.75) A_2 = 0$$

In matrix form: $$\begin{bmatrix} 3 & -1.5 \\ -0.75 & 0.75 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} - \omega^2 \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Let $\omega^2 = \lambda$, $$[A] = \begin{bmatrix} 3 & -1.5 \\ -0.75 & 0.75 \end{bmatrix} \qquad [X] = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

Then $[A]\,[X] - \lambda\,[X] = 0$ or $[A]\,[X] = \lambda\,[X]$

$\lambda$ is the eigenvalue, and [X] is the eigenvector corresponding to $\lambda$.

$\omega = \sqrt{\lambda}$ is the natural frequency corresponding to $\lambda$.

$$[A]\,[X] = \lambda\,[X]$$

$$[A]\,[X] - \lambda\,[X] = 0$$

$$[A]\,[X] - \lambda\,[I]\,[X] = 0$$

$$([A] - [\lambda\,[I])\,[X] = 0$$

$$\underbrace{\begin{pmatrix} a_{11}-\lambda & a_{12} & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{2n} \\ \dots & .. & .. \\ a_{n1} & a_{n2} & a_{nn}-\lambda \end{pmatrix}}_{([A]\,-\,\lambda\,[I])} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ .. \\ x_n \end{pmatrix}}_{[X]} = \begin{pmatrix} 0 \\ 0 \\ .. \\ 0 \end{pmatrix}$$

For a solution to exist, det ([A] – λ [I]) must be zero.

$$\det\left([A] - \lambda\;[I]\right) = 0 \quad \Rightarrow \quad \underbrace{\lambda^{n} + c_1\,\lambda^{n-1} + c_2\,\lambda^{n-2} + \dots + c_n = 0}_{\text{Characteristic equation of [A]}}$$

## Example 1

Find the eigenvalues of the physical problem discussed earlier, that is, find the eigenvalues of the matrix [A].

$$[A] = \begin{pmatrix} 3 & -1.5 \\ -0.75 & 0.75 \end{pmatrix}$$

$$[A] - \lambda\,[I] = \begin{pmatrix} 3-\lambda & -1.5 \\ -0.75 & 0.75-\lambda \end{pmatrix}$$

$$\det\left([A] - \lambda\,[I]\right) = (3-\lambda)\,(0.75-\lambda) - (-0.75)\,(-1.5) = 0$$

$$\lambda^2 - 3.75\,\lambda + 1.125 = 0 \qquad \lambda = \frac{3.75 \mp 3.092}{2}$$

$$\underbrace{\lambda_1 = 3.421 \qquad \lambda_2 = 0.3288}_{\text{Eigenvalues}}$$

**Example 2**

Find the eigenvectors of the physical problem discussed earlier, that is, find the eigenvectors of the matrix [A].

$$[A] = \begin{pmatrix} 3 & -1.5 \\ -0.75 & 0.75 \end{pmatrix}$$

For $\lambda_1 = 3.421$ :

$$\left([A] - \lambda_1 [I]\right) [X_1] = 0 \qquad \begin{pmatrix} 3 - 3.421 & -1.5 \\ -0.75 & 0.75 - 3.421 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -0.421 & -1.5 \\ -0.75 & -2.671 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Set $x_1 = C_1$, then $[X_1] = C_1 \begin{pmatrix} 1 \\ -0.2808 \end{pmatrix}$ is the eigenvector for $\lambda_1 = 3.421$

Similarly, show that the eigenvector for $\lambda_2 = 0.3288$ is $[X_2] = C_2 \begin{pmatrix} 1 \\ 1.781 \end{pmatrix}$

**Example 3**

Find the eigenvalues and eigenvectors of the matrix [A].

$$[A] = \begin{pmatrix} 1.5 & 0 & 1 \\ -0.5 & 0.5 & -0.5 \\ -0.5 & 0 & 0 \end{pmatrix}$$

Answer: $\lambda_1 = 1$, $\lambda_2 = 0.5$, $\lambda_3 = 0.5$

Note that there are repeated eigenvalues, $\lambda_2 = \lambda_3 = 0.5$. However, there should be three distinct eigenvectors.

The eigenvector for $\lambda_1 = 1$ is $$[X_1] = C_1 \begin{pmatrix} 1 \\ -0.5 \\ -0.5 \end{pmatrix}$$

Substitute for $\lambda_2 = \lambda_3 = 0.5$ and obtain $$\begin{bmatrix} 1 & 0 & 1 \\ -0.5 & 0 & -0.5 \\ -0.5 & 0 & -0.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Then set $x_1$ and $x_2$ equal to $C_1$ and $C_2$, and show that the eigenvectors for $\lambda_2 = \lambda_3 = 0.5$ are

$$[X_2] = C_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \qquad [X_3] = C_3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

## Theorems of eigenvalues and eigenvectors

1. If [A] is a n x n upper or lower triangular or diagonal matrix, the eigenvalues of [A] are the diagonal elements.
2. If [A] is a singular matrix, $\lambda = 0$ is an eigenvalue of [A].
3. [A] and $[A]^T$ have the same eigenvalues.
4. Eigenvalues of a symmetric matrix are real.
5. Eigenvectors of a symmetric matrix are orthogonal, but only for distinct eigenvalues.
6. The absolute value of det [A] is the product of the absolute values of the eigenvalues of [A].
7. …

## Example 4

Find the eigenvalues and eigenvectors of the matrix [A].

$$[A] = \begin{pmatrix} 6 & 0 & 0 & 0 \\ 7 & 3 & 0 & 0 \\ 9 & 5 & 7.5 & 0 \\ 2 & 6 & 0 & -7.2 \end{pmatrix}$$

## Example 5

Show that [A] is singular because one of the eigenvalues is zero.

$$[A] = \begin{pmatrix} 5 & 6 & 2 \\ 3 & 5 & 9 \\ 2 & 1 & -7 \end{pmatrix}$$

## Example 6

Show that the eigenvalues [A] are: $\lambda_1 = -1.547$, $\lambda_2 = 12.33$, $\lambda_3 = 4.711$.

Find the eigenvalues of [B].

Find the determinant of [A].

$$[A] = \begin{pmatrix} 2 & -3.5 & 6 \\ 3.5 & 5 & 2 \\ 8 & 1 & 8.5 \end{pmatrix} \qquad [B] = \begin{pmatrix} 2 & 3.5 & 8 \\ -3.5 & 5 & 1 \\ 6 & 2 & 8.5 \end{pmatrix}$$

## Finding eigenvalues and eigenvectors numerically – Power Method

One of the most common methods used for finding eigenvalues and eigenvectors is the power method. It is used to find the largest eigenvalue in an absolute sense.

Note that if this largest eigenvalue is repeated, this method will not work.

Also this eigenvalue needs to be distinct.

The method is as follows:

Guess $[X^0]$ to be the eigenvector with one of the elements equal to 1.

Find $[A]\ [X^0] = \lambda^0\ [X^0] = [Y^1]$

Scale $[Y]^1$ such that the chosen unity component remains unity. $[Y^1] = \lambda^1\ [X^1]$

Repeat the above steps until the eigenvalue converges with a pre-specified error.

## Power Method

Given $A X = \lambda X$ $[A - \lambda I] X = 0$

$$A A X = A \lambda X = \lambda A X = \lambda^2 X$$

$$A A A X = A \lambda^2 X = \lambda^2 A X = \lambda^3 X$$

$$A A A A \ldots A X = \lambda^n X$$

$\lambda^n$ is the $n^{th}$ power of the eigenvector that has the largest magnitude.

X is the corresponding eigenvector of A

## Example 7

Using the power method, find the largest eigenvalue and the corresponding eigenvector of the matrix [A].

$$[A] = \begin{pmatrix} 1.5 & 0 & 1 \\ -0.5 & 0.5 & -0.5 \\ -0.5 & 0 & 0 \end{pmatrix}$$

Start with $[X^0] = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\quad [A]\,[X^0] = \begin{pmatrix} 2.5 \\ -0.5 \\ -0.5 \end{pmatrix} = 2.5 \begin{pmatrix} 1 \\ -0.2 \\ -0.2 \end{pmatrix}$

$\lambda^1 = 2.5$ $\quad [X^1] = \begin{pmatrix} 1 \\ -0.2 \\ -0.2 \end{pmatrix}$ $\quad [A]\,[X^1] = \begin{pmatrix} 1.3 \\ -0.5 \\ -0.5 \end{pmatrix} = 1.3 \begin{pmatrix} 1 \\ -0.3846 \\ -0.3846 \end{pmatrix}$

| i | $\lambda^i$ | $[X^i]$ | Relative error in $\lambda^i$ |
|---|---|---|---|
| 0 | | $(1 \quad 1 \quad 1)^T$ | |
| 1 | 2.5 | $(1 \quad -0.2 \quad -0.2)^T$ | 92.307 % |
| 2 | 1.3 | $(1 \quad -0.38462 \quad -0.38462)^T$ | 16.552 % |
| 3 | 1.1154 | $(1 \quad -0.44827 \quad -0.44827)^T$ | 6.0529 % |
| 4 | 1.0517 | $(1 \quad -0.47541 \quad -0.47541)^T$ | 1.2441 % |
| 5 | 1.02459 | $(1 \quad -0.48800 \quad -0.48800)^T$ | |

Exact solution is: $\lambda = 1$ and $[X] = \begin{pmatrix} 1 \\ -0.5 \\ -0.5 \end{pmatrix}$

Note that when the maximum magnitude eigenvector is repeated, one with a plus sign and another with a minus sign, the power method will give you oscillations.

Read about **power method with shifts** which would theotically give you all the eigenvalues and the corresponding eigenvectors if you know how to shift, i.e., use the proper shifting factors.

## Power Method with Shifts

The eigenvalue of a system $A\,X = \lambda\,X$ closest to a given scalar s can be found.

$s\,I\,X = s\,X$

$A\,X - s\,X = \lambda\,X - s\,X$

$[A - s\,I]\,X = (\lambda - s)\,X$ $[A - s\,I]$ is the shifted matrix

$(\lambda - s)$ is the eigenvector of the shifted matrix.

The eigenvalues of the inverse matrix $A^{-1}$ are $1/\lambda$'s of A. So, one may find the smallest magnitude eigenvalue and corresponding eigenvector of A.

One may use the shifiting with the inverse matrix as well. RTB