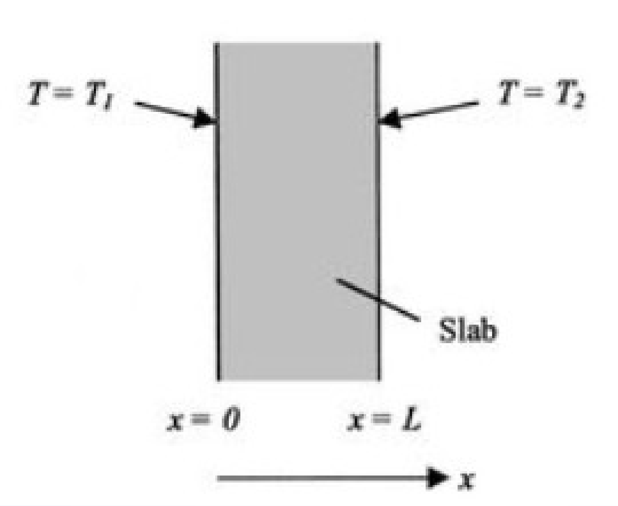
3. Steady, One-dimensional Heat Conduction

Steady state: Temperature does not vary with time.

One dimensional: Temperature is a function of one dimension only, such as x.

3.1 Slab with constant properties, such as k



Differential equation with heat generation:

$$\frac{d}{dx}\left(k\,\frac{dT}{dx}\right) + \dot{e}_{gen} = 0$$

Note that the partial differentials no longer exist which makes the solution simple.

The heat generation term can be a constant or a function of position (space variable) x.

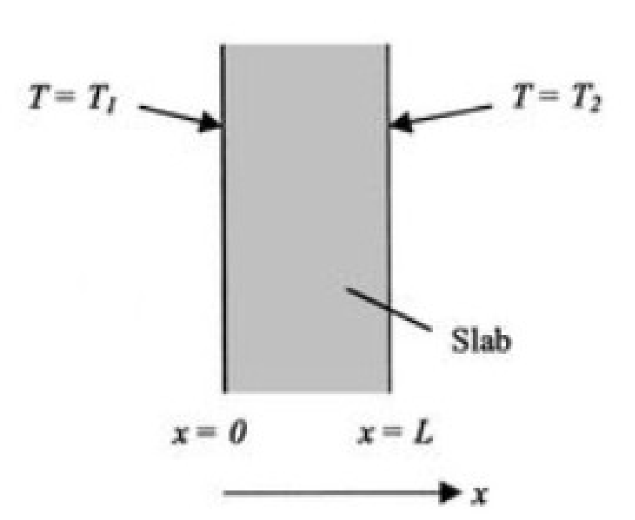
The solution depends on the BC's that are used to to find the const. of integration.

Differential equation without heat generation:

$$\frac{d}{dx}\left(k\,\frac{dT}{dx}\right) = 0$$

$$\frac{d^2T}{dx^2} = 0 \quad \text{for constant k}$$

Example 1



Determine T(x) and q for a slab with boundary surfaces at x = 0 and x = L are kept at uniform temperatures T_1 and T_2, respectively (no heat generation).

$$\frac{d^2T}{dx^2} = 0 \quad in \quad 0 \le x \le L \qquad T(x) = C_1\,x + C_2$$

BC's: at x = 0 T(0) = T_1 => $C_2 = T_1$

at x = L T(L) = T_2 => $C_1 = \dfrac{T_2 - T_1}{L}$

$$T(x) = \frac{T_2 - T_1}{L}\,x + T_1$$

Heat flux: $\dot{q} = - k \dfrac{dT}{dx} = k \left(\dfrac{T_1 - T_2}{L} \right)$ Independent of x

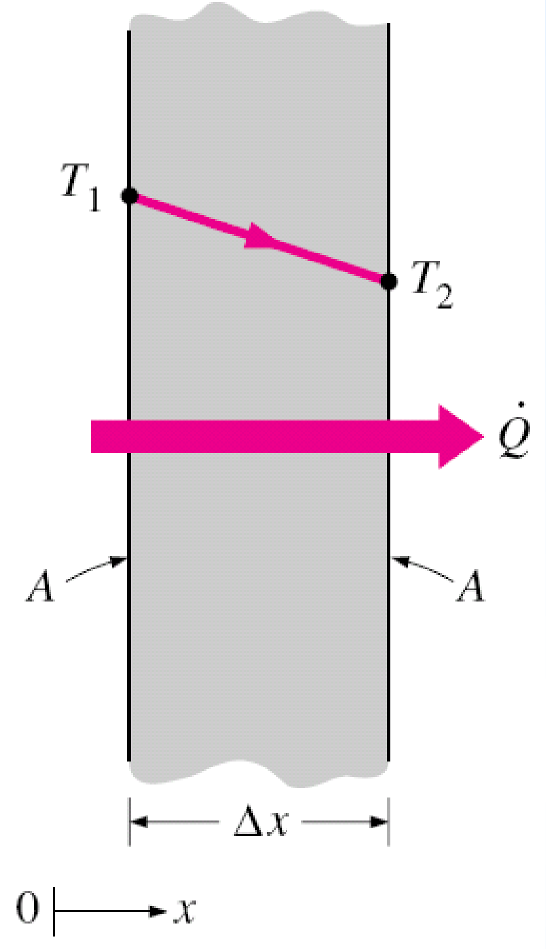
Heat flow rate: $\dot{Q} = \dot{q}\, A = k\, A \left(\dfrac{T_1 - T_2}{L} \right)$ Independent of x

If there is no heat generation, heat flux q and heat flow rate Q are independent of position (x) (any geometry). Therefore, q and Q can be found by solving the Fourier equation only, taking q and/or Q constant (not a function of x).

Re-solve the Example

$$\dot{q} = - k \frac{dT}{dx} \quad \Rightarrow \quad \dot{q}\, dx = - k\, dT \quad \Rightarrow \quad \int_0^L \dot{q}\, dx = \int_{T_1}^{T_2} - dT$$

$$\dot{q} \int_0^L dx = \int_{T_1}^{T_2} - dT \quad \Rightarrow \quad \dot{q} = k \left(\frac{T_1 - T_2}{L} \right) \quad \Rightarrow \quad \boxed{\dot{Q} = \frac{T_1 - T_2}{\dfrac{L}{k\, A}}}$$



Temperature profile: $T(x) = \frac{T_2 - T_1}{L} x + T_1$

Heat flux $\dot{q} = k \left(\frac{T_1 - T_2}{L} \right)$

Heat flow rate: $\dot{Q} = k A \frac{T_1 - T_2}{L} = \frac{T_1 - T_2}{\frac{L}{k A}}$

Example 2

Determine T(x) and q(x) for a slab with uniform heat generation, e_{gen} in W/m³. The boundary at surface x = 0 is kept at a uniform temperature T_1 and the boundary at x = L dissipates heat by convection into an environment at a constant temperature T_∞ with convective heat transfer coefficient h.

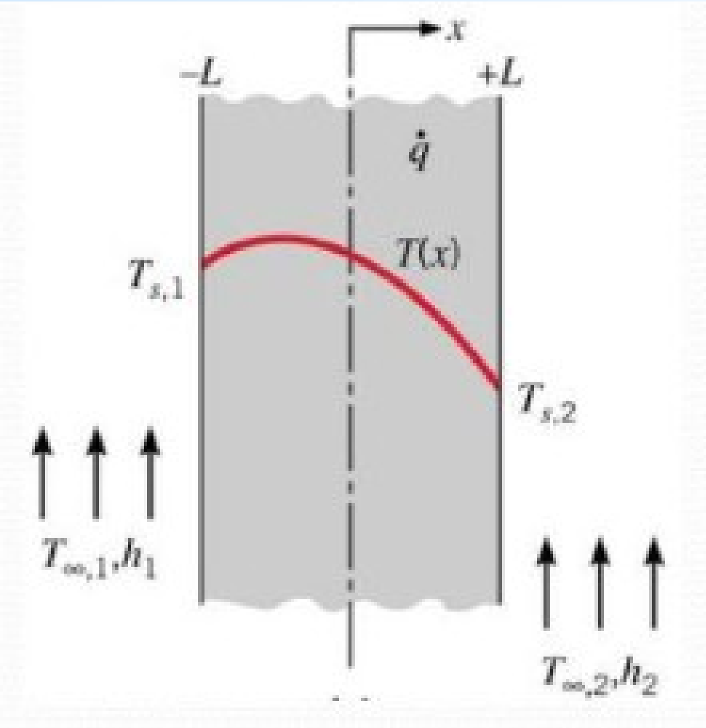
Differential equation: $$\frac{d^2T}{dx^2} + \frac{\dot{e}_{gen}}{k} = 0 \quad \text{in} \quad 0 \le x \le L$$

Boundary conditions:

1. $T(0) = T_1$ at $x = 0$
2. $k \frac{dT}{dx} + h\,T(x) = h\,T_\infty$ at $x = L$

Solution: $$\frac{dT}{dx} = -\frac{\dot{e}_{gen}}{k} x + C_1$$

$$T(x) = -\frac{\dot{e}_{gen}}{2\,k} x^2 + C_1 x + C_2$$



Apply Boundary conditions:

From (1): $C_2 = T_1$

From (2): $$C_1 = \frac{(T_\infty - T_1)\, h}{k + h\, L} + \frac{\dot{e}_{gen}\, L\, (2\, k + h\, L)}{2\, k\, (k + h\, L)}$$

This is a parabolic distribution

$$T(x) = -\frac{\dot{e}_{gen}}{2\, k}\, x^2 + \left[\frac{(T_\infty - T_1)\, h}{k + h\, L} + \frac{\dot{e}_{gen}\, L\, (2\, k + h\, L)}{2\, k\, (k + h\, L)}\right] x + T_1$$

Define a non-dimensional parameter called Biot number: $Bi = \dfrac{h\,L}{k}$

$$T(x) - T_1 = \left(\frac{T_\infty - T_1}{1 + 1/Bi}\right)\frac{x}{L} + \frac{\dot{e}_{gen}\,L^2}{2\,k}\left[\left(\frac{1 + 2/Bi}{1 + 1/Bi}\right)\frac{x}{L} - \left(\frac{x}{L}\right)^2\right]$$

We shall return to the importance of the Biot number later on.

Heat flux: $$q(x) = -k\,\frac{dT}{dx} = \dot{e}_{gen}\,x - C_1\,k$$ Heat flux is a function of x

Jean-Baptiste Biot

French Scientist

1774 - 1862

Note the following: When there is heat generation in the medium, the heat flux varies with position.

For most practical problems, the amount of heat flow at the boundaries is of interest.

Examine the cases when Bi -> ∞ and when Bi -> 0:

When Bi -> ∞ then h -> ∞ and the boundary condition reduces to $T(L) = T_{\infty}$

When Bi -> 0 then h -> 0 and the boundary condition reduces to $dT/dx = 0$.

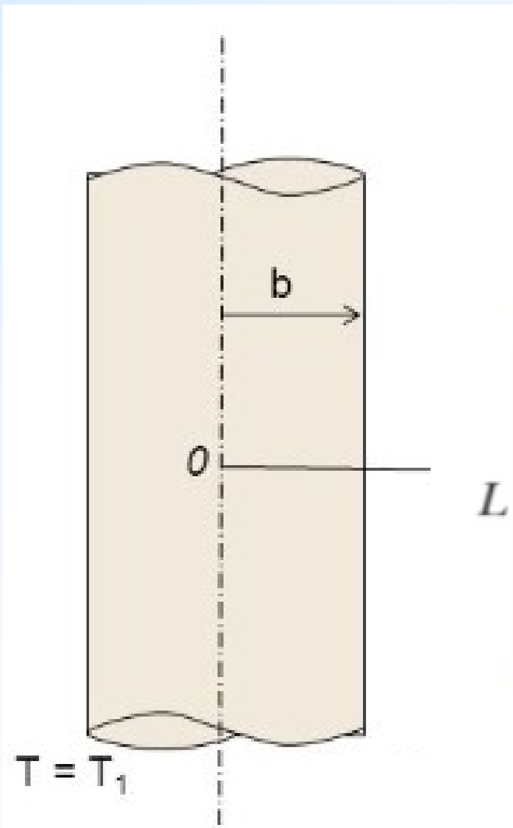
That means that there is no heat flow through that surface at x = 0.

Example 3

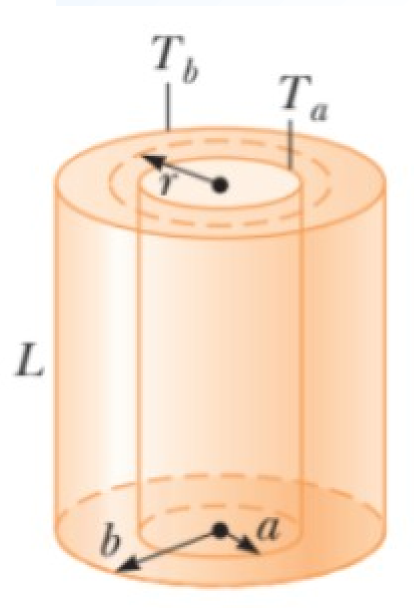
Consider an infinite plane wall (slab) 2 L thick, in which there is uniform volumetric heat generation rate, q'''. The wall surfaces are maintained at $T = T_1$ at $x = -L$ and $T = T_2$ at $x = +L$. For constant thermal conductivity k, steady-state operating conditions, and defining the origin of the x coordinate from the centerline of the plane, show that the solution of the general conduction equation for the temperature distribution (profile) in the wall is

$$T(x) = \frac{q''' L^2}{2\,k}\left[1 - \left(\frac{x}{L}\right)^2\right] + \frac{T_2 - T_1}{2}\left(\frac{x}{L}\right) + \frac{T_1 + T_2}{2}$$

3.2 Cylinder with constant properties, such as k



Solid cyclinder



Hollow cyclinder

Differential equation

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{\dot{e}_{gen}}{k}=0$$

Heat flux $\dot{q} = -k\frac{dT(r)}{dr}$

If positive, heat flow is in the positive r direction

For a solid cylinder, the solution of the differential equation requires two boundary conditions at r = 0 and at r = b.

The boundary condition at r = b can be of any kind. For steady state , it cannot be an insulated surface BC. (Why?)

The boundary condition at the centre (r = 0) is implicit. It is known as the symmetry condition. (Symmetry of what?):

$$\text{At } r = 0 \quad \frac{dT}{dr} = 0$$

or T is finite

Example 4

Determine T(r) and q(b) for a solid cyclinder with uniform heat generation, e_{gen} in W/m^3. The boundary at surface r = b is kept at a uniform temperature T_1.

Differential equation: $$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{\dot{e}_{gen}}{k}=0 \quad \text{in} \quad 0 \le r \le b$$

Boundary conditions:
1. $T(b) = T_1$ at $r = b$
2. $\frac{dT}{dr} = 0$ at $r = 0$

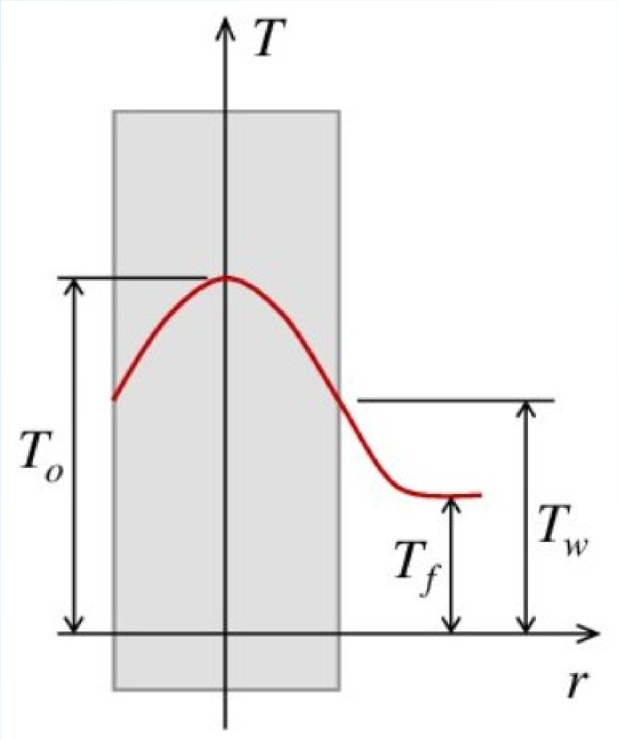
Solution: $$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = -\frac{\dot{e}_{gen}}{k}\,r \quad \Rightarrow \quad r\frac{dT}{dr} = -\frac{\dot{e}_{gen}}{k}\,r^2 + C_1$$

$$\frac{dT}{dr} = -\frac{\dot{e}_{gen}}{2\,k}\,r + \frac{C_1}{r} \quad \Rightarrow \quad T(r) = -\frac{\dot{e}_{gen}}{4\,k}\,r^2 + C_1\ln(r) + C_2$$

Apply Boundary conditions:

From (2): $C_1 = 0$

From (1): $C_2 = T_1 + \frac{\dot{e}_{gen}}{4\,k}\,b^2$



Temperature profile has a parabolic distribution:

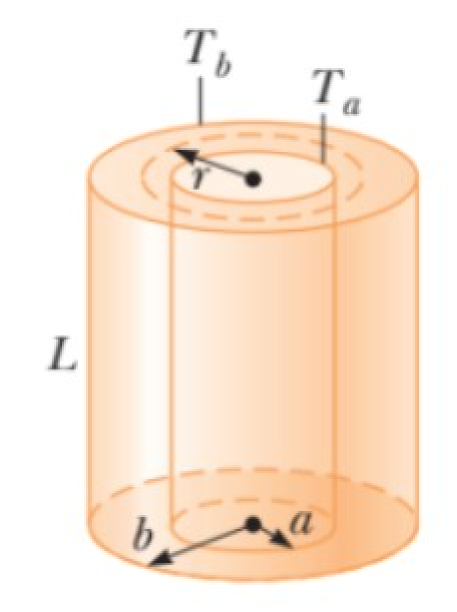
$$T(r) = -\frac{\dot{e}_{gen}}{4\,k}\,r^2 + \frac{\dot{e}_{gen}}{4\,k}\,b^2 + T_w$$

Heat flux at the surface r = b:

$$q(r)\Big|_{r=b} = -k\,\frac{dT}{dr}\Big|_{r=b} = -k\left(-\frac{\dot{e}_{gen}}{2\,k}\,b\right) = \frac{\dot{e}_{gen}\,b}{2}$$

Example 5

Determine T(r) and the radial heat flow rate Q for a length L in a hollow cylinder with constant rate of heat generation, e_{gen} when the boundary at surfaces at r = a and r = b are kept at uniform temperatures T_a and T_b, respectively.



Differential equation:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{\dot{e}_{gen}}{k} = 0 \quad \text{in} \quad a \le r \le b$$

Boundary conditions:
1. $T(a) = T_a$ at $r = a$
2. $T(b) = T_b$ at $r = b$

Solution:

$$T(r) = -\frac{\dot{e}_{gen}}{4\,k} r^2 + C_1 \ln(r) + C_2$$

Apply boundary conditions:

$$T(r) = -\frac{\dot{e}_{gen}}{4\,k}\,r^2 + \frac{\left(T_b - T_a\right) + \dfrac{\dot{e}_{gen}}{4\,k}\left(b^2 - a^2\right)}{\ln\left(\dfrac{b}{a}\right)}\,\ln(r) +$$

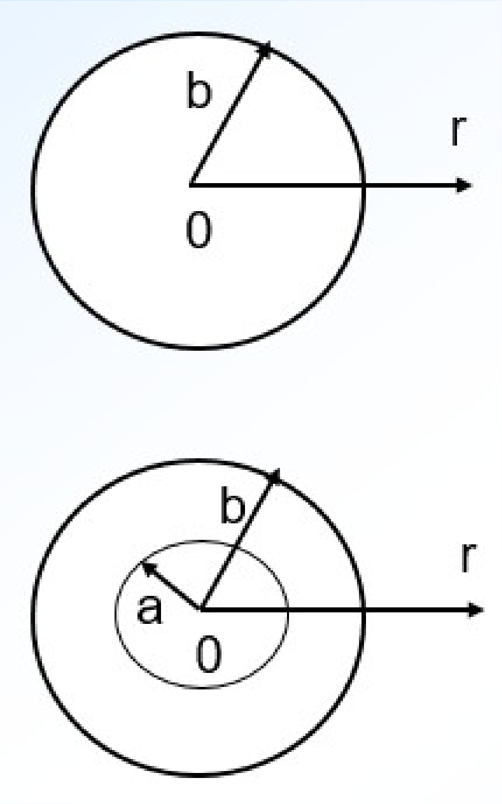
$$\left(T_a + \frac{\dot{e}_{gen}\,a^2}{4\,k}\right) - \left[\left(T_b - T_a\right) + \frac{\dot{e}_{gen}}{4\,k}\left(b^2 - a^2\right)\right]\frac{\ln(a)}{\ln\left(\dfrac{b}{a}\right)}$$

The radial heat flow rate at any position r through the cylinder for a length of H is:

$$Q(r) = q(r)\,(\text{Area}) = -k\,\frac{dT}{dr}\,(2\,\pi\,r\,L) = 2\,\pi\,L\left(\frac{\dot{e}_{gen}}{2}\,r^2 - C_1\,k\right)$$

Show that if there is no heat generation, the heat flow rate is independent of the radial position r.

3.3 Sphere with constant properties, such as k



Differential equation $$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right)+\frac{\dot{e}_{gen}}{k}=0$$

Heat flux $$\dot{q}=-k\,\frac{dT(r)}{dr}$$

The considerations for the boundary conditions (such as symmetry at r = 0) is the same as that of a cylinder.

Example 6

Determine T(r) and the total radial heat flow rate Q in a hollow sphere when the surfaces at r = a and r = b are kept at uniform temperatures T_a and T_b, respectively. (No heat generation.)

Differential equation: $$\frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) = 0 \quad \text{in} \quad a \le r \le b$$

Boundary conditions:

1. $T(a) = T_a$ at $r = a$
2. $T(b) = T_b$ at $r = b$

Solution: $$\frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) = 0 \quad \Rightarrow \quad r^2 \frac{dT}{dr} = C_1 \quad \Rightarrow \quad T(r) = -\frac{C_1}{r} + C_2$$

Apply boundary conditions: $$T_a = -\frac{C_1}{a} + C_2 \qquad T_b = -\frac{C_1}{b} + C_2$$

Apply boundary conditions: $C_1 = \frac{a\,b}{a - b}\left(T_a - T_b\right)$ $C_2 = \frac{b\,T_b - a\,T_a}{b - a}$

Temperatute profile: $T(r) = \frac{1}{b - a}\left[a\,T_a\left(\frac{b}{r} - 1\right) + b\,T_b\left(1 - \frac{a}{r}\right)\right]$

Radial heat flow rate: $\dot{Q} = \dot{q}(r)\,(\text{Area}) = -k\,\frac{dT}{dr}\left(4\,\pi\,r^2\right) = -4\,\pi\,k\,C_1$

$$\dot{Q} = 4\,\pi\,k\,\frac{a\,b}{a - b}\left(T_a - T_b\right)$$

Q is independent of position, i.e., it can be found with Fourier equation only, as before.

3.4 The Concept of Thermal Resistance

The total heat flow rate through a solid can be related to thermal resistance if these assumptions are true:

- One-dimensional, steady-state heat conduction;
- Finite regions;
- No heat generation;
- Constant thermal conductivity; and
- Prescribed temperatures at the boundaries.

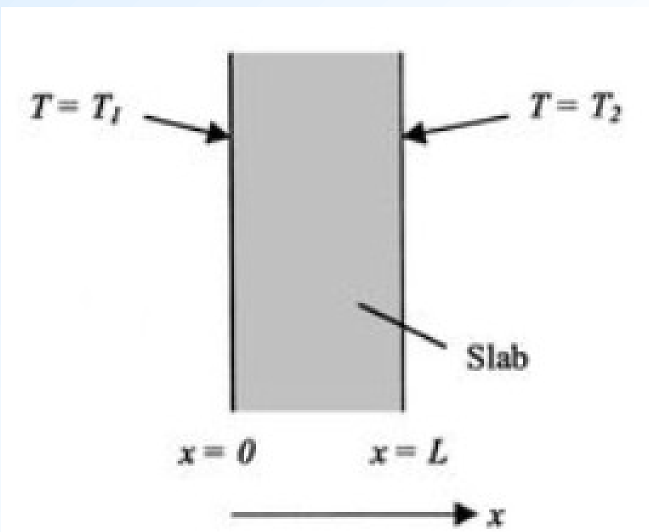
If so, then $\dot{Q} = \dfrac{\Delta T}{R}$ where

- ΔT: Difference between temperatures at the boundaries
- R: Thermal resitance in $^\circ C / W$

The thermal resistance is analogous to electrical resistance defined by:

$$\text{Current} = \frac{\text{Electric potential difference}}{\text{Electric resistance}} \qquad I = \frac{\Delta V}{R} \qquad \text{Similarly:} \quad Q = \frac{\Delta T}{R}$$

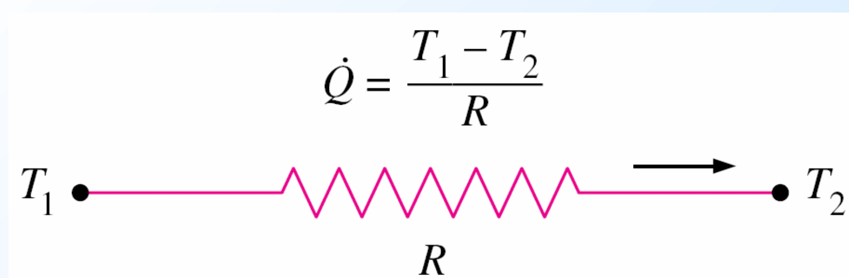
3.4.1 For the Slab



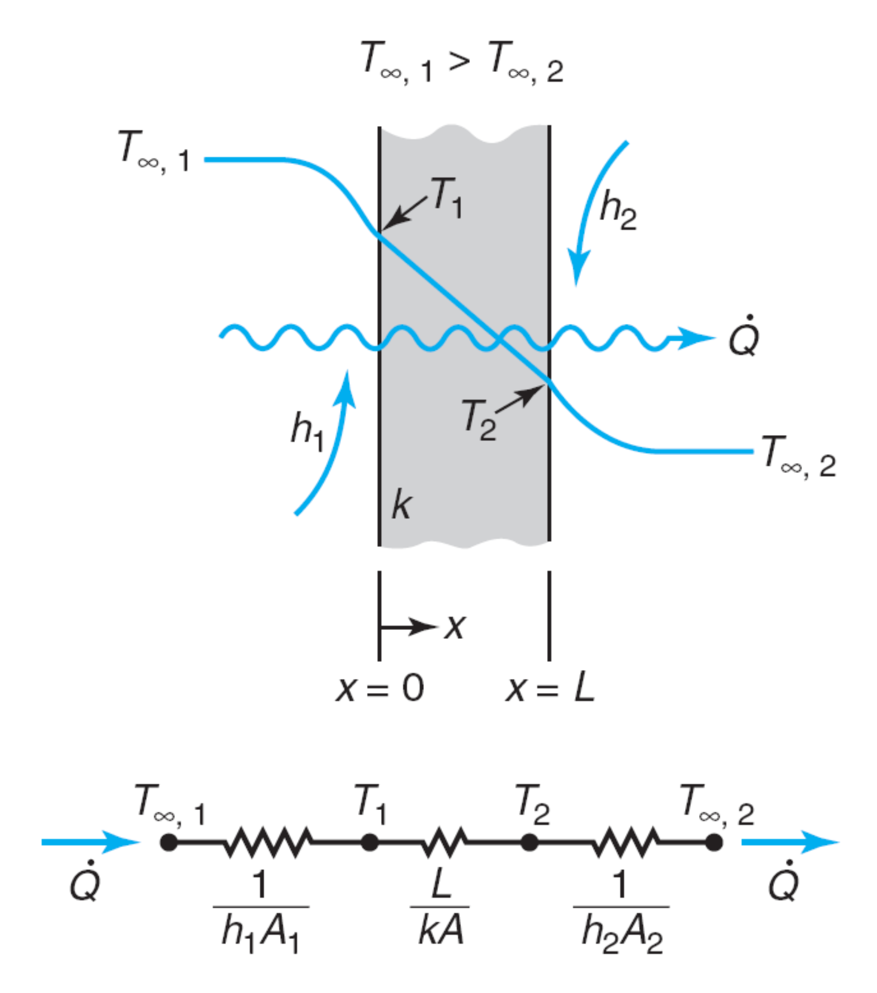
From Example 1

$$\dot{Q} = A\,k\,\frac{T_1 - T_2}{L} = \frac{T_1 - T_2}{\dfrac{L}{A\,k}}$$

$$\boxed{R_{slab} = \frac{L}{A\,k}}$$



Slab with convective boundaries



$$\dot{Q} = A\,h_1\left(T_{\infty,1} - T_1\right) = \frac{T_{\infty,1} - T_1}{\dfrac{1}{A\,h_1}}$$

$$\dot{Q} = A\,k\left(\frac{T_1 - T_2}{L}\right) = \frac{T_1 - T_2}{\dfrac{L}{A\,k}}$$

$$\dot{Q} = A\,h_2\left(T_2 - T_{\infty,2}\right) = \frac{T_2 - T_{\infty,2}}{\dfrac{1}{A\,h_2}}$$

Note that it is the same Q.

Example 7

A furnace wall 200 mm thick is made of a material having thermal conductivity of 1.45 W/m.K. The inner and outer surface are exposed to average temperatures of 3500 °C and 400 °C, respectively. If the gas and air film coefficients are 58 and 11.63 W/m^2.K, respectively, find the rate of heat transfer through a wall of 2.5 square meters. Also, find the temperatures on the two sides of the wall.

$$\dot{Q} = \frac{T_{\infty,1} - T_{\infty,2}}{\frac{1}{A\,h_1} + \frac{L}{A\,k} + \frac{1}{A\,h_2}} = \frac{350 - 40}{\frac{1}{(2)\,(58)} + \frac{0.2}{(2)\,(1.45)} + \frac{1}{(2)\,(11.63)}}$$

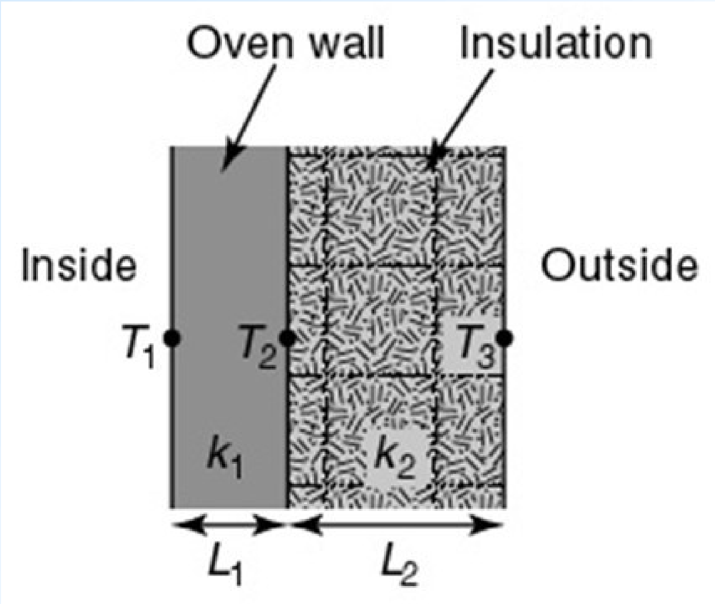
$$= 3214 \text{ W}$$

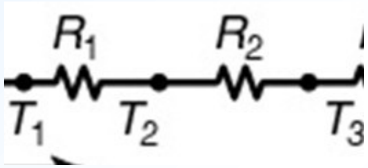
$$\dot{Q} = 3214 \text{ W} = A\,h_1\,(T_{\infty,1} - T_1) = (2)\,(58)\,(350 - T_1)$$

$$= A\,h_2\,(T_2 - T_{\infty,2}) = (2)\,(11.63)\,(T_2 - 40)$$

$T_1 = 327.84$ °C

$T_2 = 150.5$ °C





Steady, one-dim. heat conduction through multi-layered slabs:

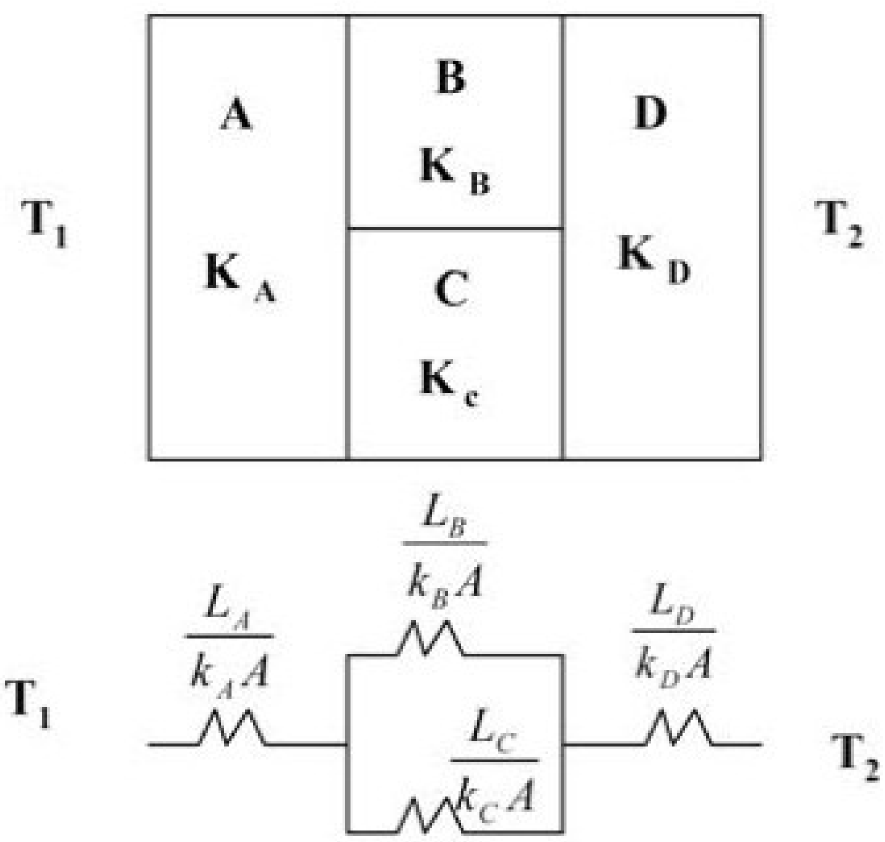
- $$\dot{Q}_{cond} = k_1 A \frac{T_1 - T_2}{L_1} = k_2 A \frac{T_2 - T_3}{L_2}$$

- $$\dot{Q}_{cond} = \frac{T_1 - T_2}{L_1/k_1 A} = \frac{T_2 - T_3}{L_2/k_2 A}$$

- $$\dot{Q}_{cond} = \frac{T_1 - T_2}{R_1} = \frac{T_2 - T_3}{R_2} = \frac{T_1 - T_3}{R_1 + R_2}$$

- $$R_{cond} = R_1 + R_2 = \frac{L_1}{k_1 A} + \frac{L_2}{k_2 A}$$

Steady, one-dimensional heat conduction through multi-layered slabs:

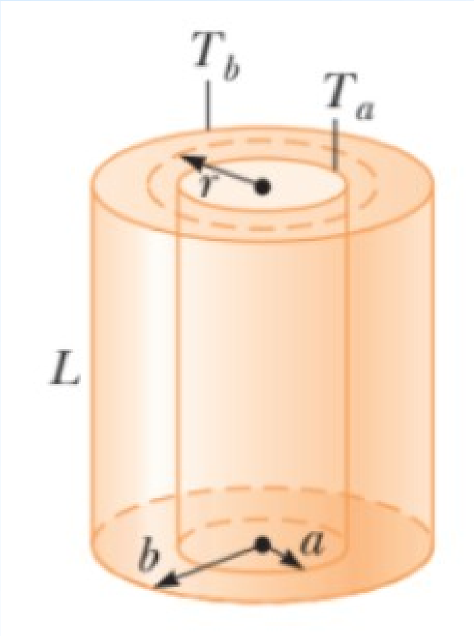


$A_B + A_C = A_A = A_D$

$L_B = L_C$

- $$\dot{Q}_{cond} = \frac{T_1 - T_2}{R_{total}}$$
- $$R_{total} = R_A + \frac{1}{\frac{1}{R_B} + \frac{1}{R_C}} + R_D$$

3.4.2 For the Hollow Cylinder



From Example 4 $$\dot{Q} = 2\pi L \left(\frac{\dot{e}_{gen}}{2} r^2 - C_1 k \right)$$

$$\dot{e}_{gen} = 0 \quad \Rightarrow \quad \dot{Q} = 2\pi L\,(-C_1 k)$$

$$C_1 = \frac{T_b - T_a}{\ln\left(\frac{b}{a}\right)} \quad \Rightarrow \quad \dot{Q} = \frac{2\pi L}{\ln\left(\frac{b}{a}\right)} (T_a - T_b)$$

$$R_{cyl} = \frac{\ln\left(\frac{b}{a}\right)}{2\pi L k}$$

This can be re-arranged to take a similar form as that of a slab:

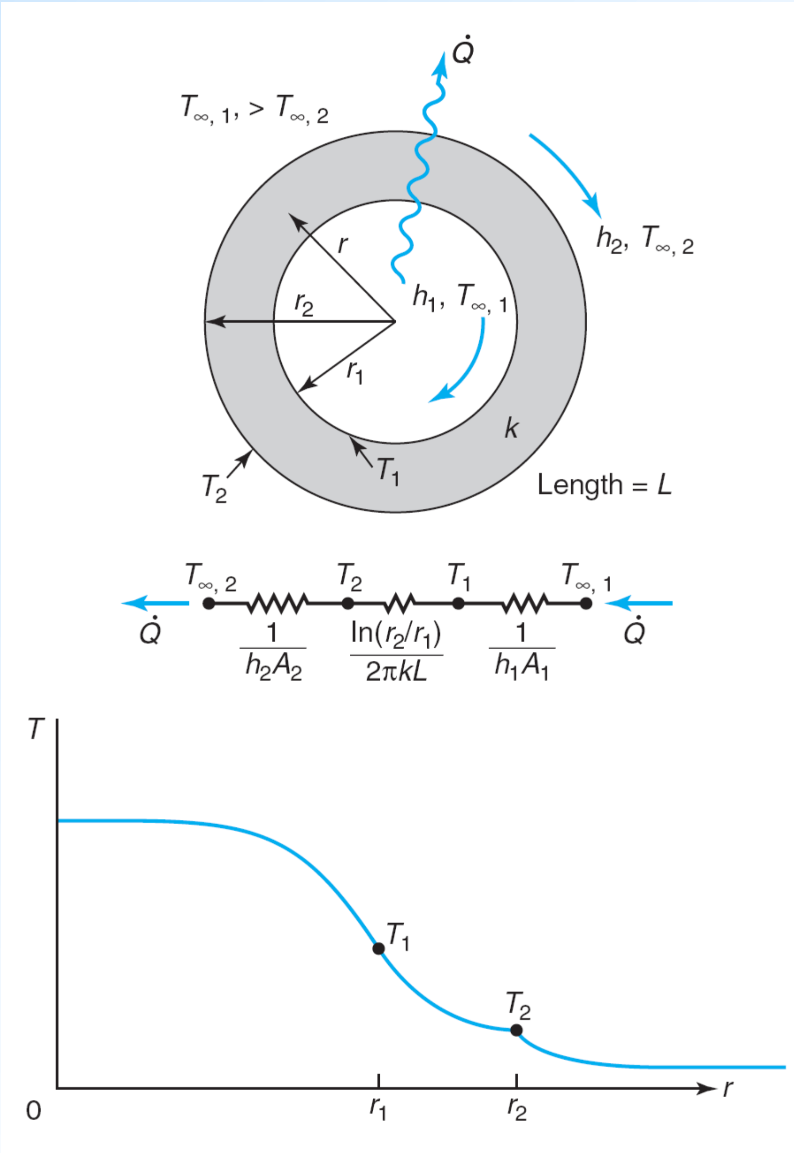
$$R_{cyl} = \frac{L_{cyl}}{A_{cyl}\ k}$$

Where: $L_{cyc} = b - a$ Thickness of the cylinder

$$A_{cyl} = \frac{A_b - A_a}{\ln\left(\frac{A_b}{A_a}\right)}$$ Logarithmic mean area

$$A_b = 2\ \pi\ b\ L$$

$$A_a = 2\ \pi\ a\ L$$



Hollow cylinder with convective boundaries

$$\dot{Q} = A_1\, h_1\, \left(T_{\infty,1} - T_1\right) = \frac{T_{\infty,1} - T_1}{\dfrac{1}{A_1\, h_1}}$$

$$\dot{Q} = 2\,\pi\, k\, L \left(\frac{T_1 - T_2}{\ln\left(r_2 / r_1\right)} \right) = \frac{T_1 - T_2}{\dfrac{\ln\left(r_2 / r_1\right)}{2\,\pi\, k\, L}}$$

$$\dot{Q} = A_2\, h_2\, \left(T_2 - T_{\infty,2}\right) = \frac{T_2 - T_{\infty,2}}{\dfrac{1}{A_2\, h_2}}$$

Note that it is the same Q.

Example 8

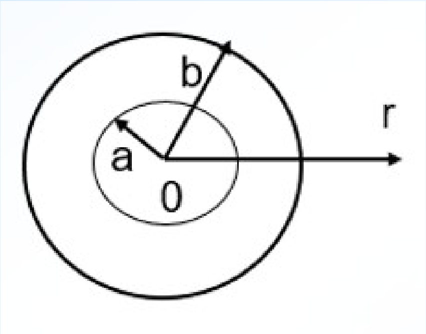
A steam pipe of inner diameter 200 mm is covered with 50 mm thick high insulated material of thermal conductivity k = 0.01 W/m.°C. The inner and outer surface temperatures maintained at 5000 °C and 1000 °C, respectively. Calculate the total heat loss per meter length of pipe?

$$r_1 = \frac{200}{2} = 100 \text{ mm}$$

$$r_2 = \frac{200 + 100}{2} = 150 \text{ mm}$$

$$\dot{Q} = \frac{T_1 - T_2}{\frac{\ln\left(r_2/r_1\right)}{2\pi k L}} = \frac{500 - 100}{\frac{\ln\left(0.15/0.1\right)}{2\pi (0.01)(1)}} = 61.98 \text{ W}$$

3.4.3 For the Hollow Sphere



From Example 5

$$\dot{Q} = 4\,\pi\,k\,\frac{a\,b}{b - a}\left(T_a - T_b\right)$$

$$R_{sph} = \frac{1}{4\,\pi\,k}\,\frac{b - a}{a\,b}$$

This can ve re-arranged:

$$R_{sph} = \frac{L_{sph}}{A_g\;k}$$

Where: $L_{sph} = b - a$ Thickness of the sphere

$A_g = \sqrt{A_a\;A_b}$ Geometric mean area

$A_b = 4\,\pi\,b^2$ $A_a = 4\,\pi\,a^2$

Example 9

A spherical shaped vessel of 1.2 m diameter is 100 mm thick. Find the rate of heat leakage, if the temperature difference between the inner and outer surface is 200 °C. Thermal conductivity of the material of sphere is 0.3 kJ/h.m.°C.

$$b = \frac{1.2}{2} = 0.6 \text{ m}$$

$$a = \frac{1.2 - (2)(0.1)}{2} = 0.5 \text{ m}$$

$$\dot{Q} = 4 \pi k \frac{a\,b}{b - a} \left(T_a - T_b\right)$$

$$= 4 \pi (0.3) \frac{(0.5)(0.6)}{0.6 - 0.5} \left(200\right) = 2262 \text{ kJ/h}$$

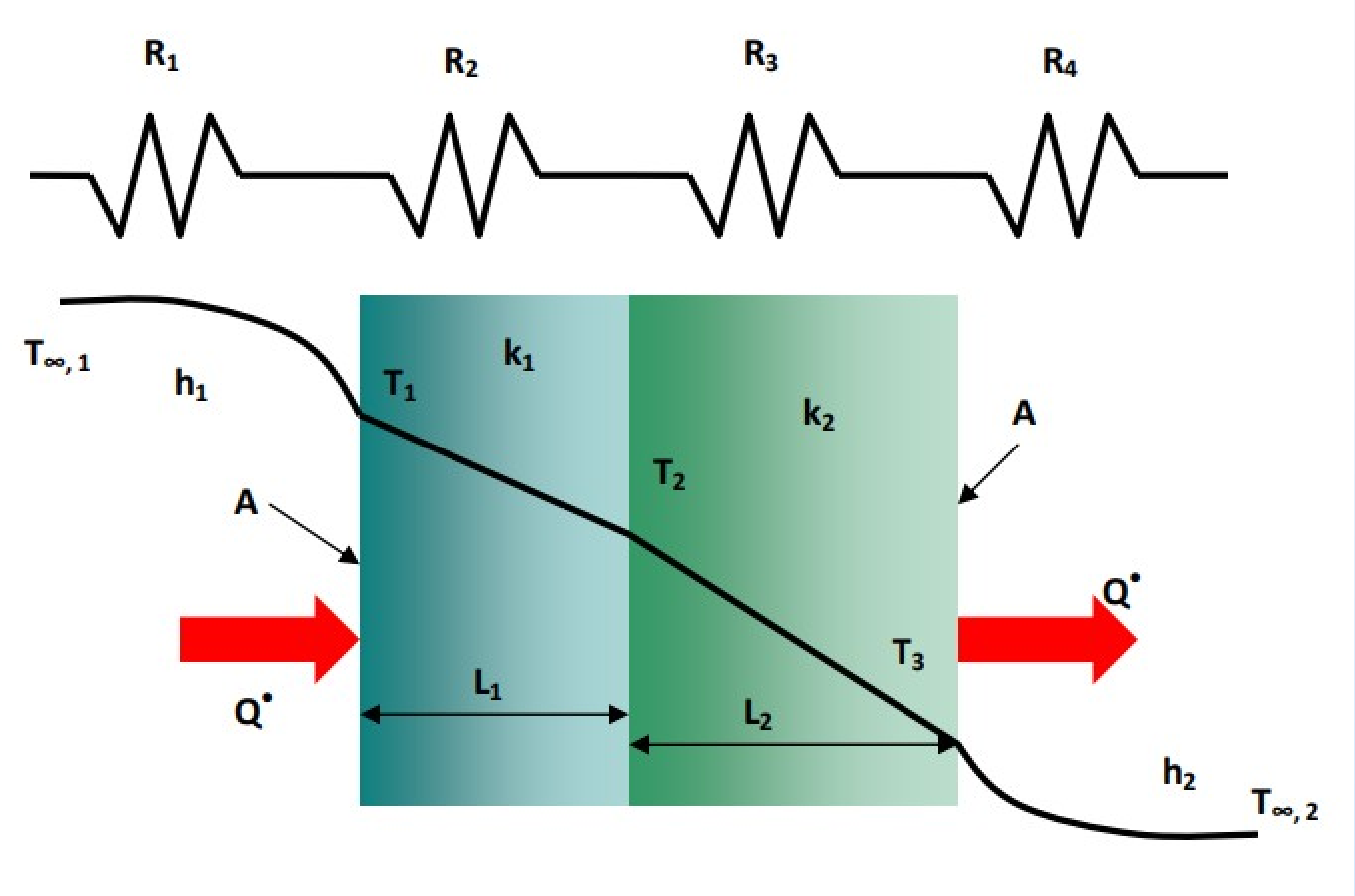
3.5 Composite Medium

If heat transfer takes place through a medium composed of several different layers, connected in parallel or in series, with the same or different thermal conductivities, the rate of heat flow can be calculated using the thermal resistance concept.

In addition to the previous assumptions, add the following:

- Perfect thermal contact (no temperature drop at the interfaces of the layers);
- Interior and exterior surafces of the structures are subjected to convective heat transfer to fluids at constant mean temepratures $T_{\infty,1}$ and $T_{\infty,2}$, and with convective heat transfer coefficients h_1 and h_2.

3.5.1 Slabs Connected in Parallel



The heat transfer rate, Q, through an area A of this composite structure is the same through each layer.

$$\dot{Q} = A\,h_1\left(T_{\infty,1} - T_1\right) = A\,k_1\,\frac{T_1 - T_2}{L_1} = A\,k_2\,\frac{T_2 - T_3}{L_2} = A\,h_2\left(T_3 - T_{\infty,2}\right)$$

In terms of thermal resistances:

$$\dot{Q} = \frac{T_{\infty,1} - T_1}{\dfrac{1}{A\,h_1}} = \frac{T_1 - T_2}{\dfrac{L_1}{A\,k_1}} = \frac{T_2 - T_3}{\dfrac{L_2}{A\,k_2}} = \frac{T_3 - T_{\infty,2}}{\dfrac{1}{A\,h_2}}$$

$$\dot{Q} = \frac{T_{\infty,1} - T_1}{R_1} = \frac{T_1 - T_2}{R_2} = \frac{T_2 - T_3}{R_3} = \frac{T_3 - T_{\infty,2}}{R_4}$$

Define total resistance, $R = R_1 + R_2 + R_3 + R_4$

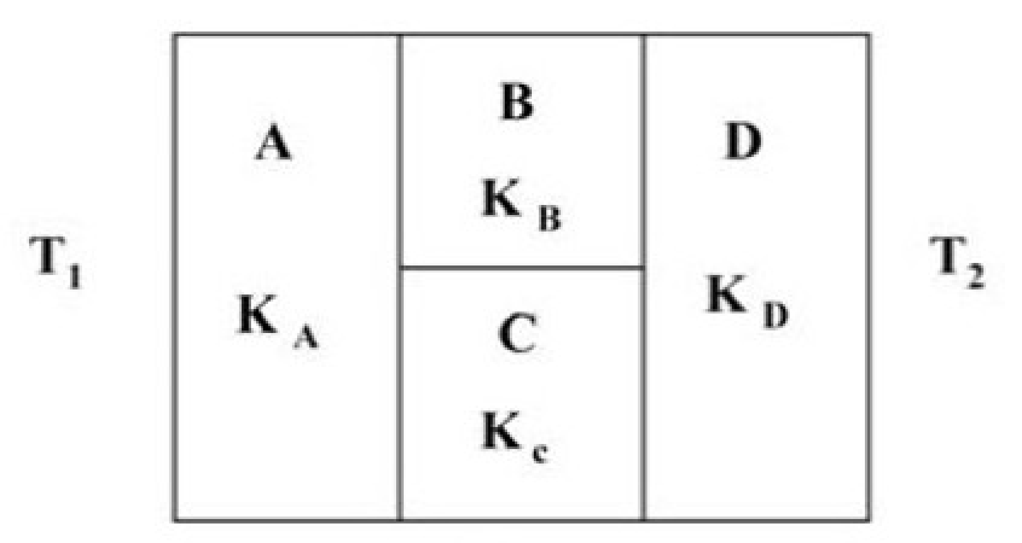
$$\dot{Q} = \frac{T_{\infty,1} - T_1}{R_1} = \frac{T_1 - T_2}{R_2} = \frac{T_2 - T_3}{R_3} = \frac{T_3 - T_{\infty,2}}{R_4} = \frac{T_{\infty,1} - T_{\infty,2}}{R_{total}}$$

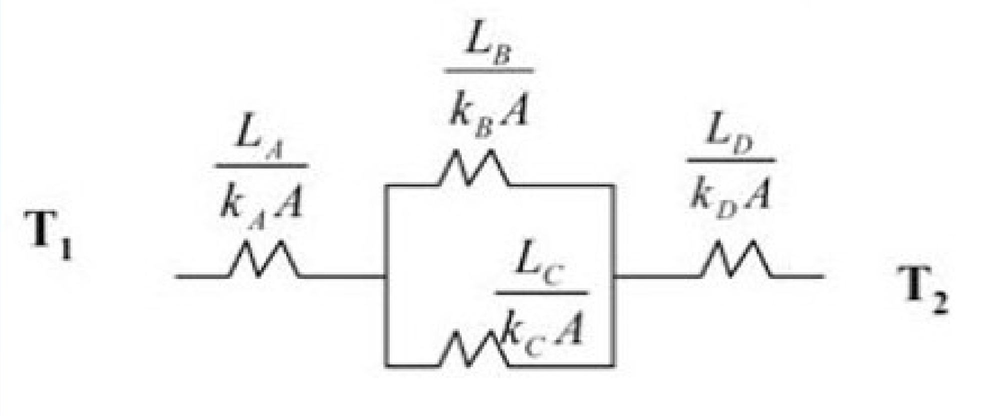
Define **Overall Heat Transfer Coefficient**, U:

$$\dot{Q} = A\,U\left(T_{\infty,1} - T_{\infty,2}\right) = \frac{T_{\infty,1} - T_{\infty,2}}{\dfrac{1}{A\,U}} = \frac{T_{\infty,1} - T_{\infty,2}}{R_{total}}$$

$$U = \frac{1}{A\,R_{total}} = \frac{1}{\dfrac{1}{h_1} + \dfrac{L_1}{k_1} + \dfrac{L_2}{k_2} + \dfrac{1}{h_2}}$$

3.5.2 Slabs Connected in Series





$$L_B = L_C$$

$$A_A = A_B + A_C = A_D$$

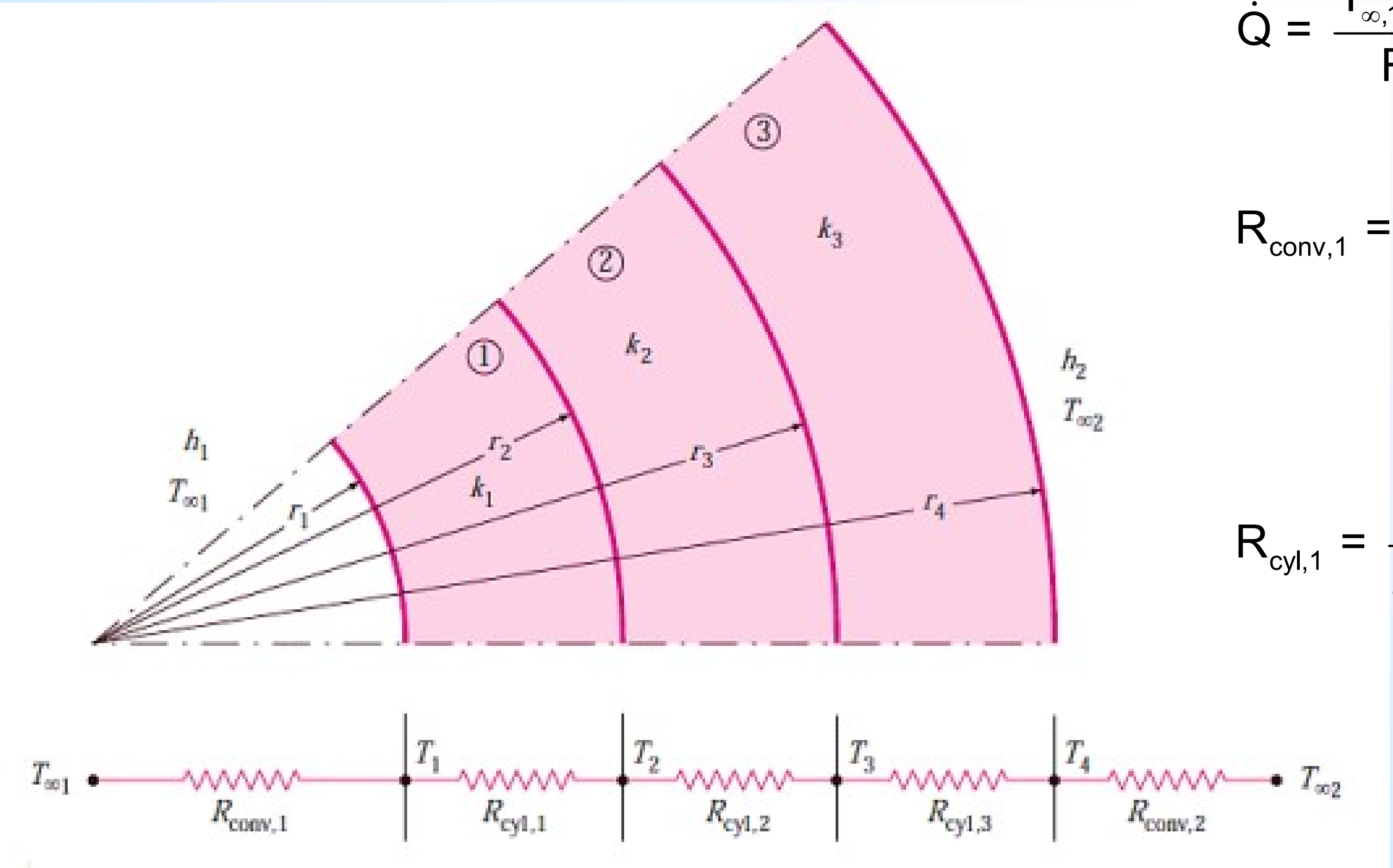
$$\dot{Q} = \frac{T_1 - T_2}{R_{total}}$$

$$R_{total} = R_A + \frac{1}{\frac{1}{R_B} + \frac{1}{R_C}} + R_D$$

Example 10

A composite wall consists of three layers of thicknesses 300 mm, 200 mm and 100 mm with thermal conductivities 1.5, 3.5, and is W/m.K, respectively. The inside surface is exposed to gases at 1200 °C with convection heat transfer coefficient as 30 $W/m^2.K$. The temperature of air on the other side of the wall is 30 °C with convective heat transfer coefficient 10 $W/m^2.K$. If the temperature at the outside surface of the wall is 180 °C, calculate the temperature at other surface of the wall, the rate of heat transfer, and the overall heat transfer coefficient.

3.5.3 Coaxial Cylinders



$$\dot{Q} = \frac{T_{\infty,1} - T_{\infty,2}}{R_{total}}$$

$$R_{conv,1} = \frac{1}{2\,\pi\,r_1\,L\,h_1}$$

$$R_{cyl,1} = \frac{\ln\left(\frac{r_2}{r_1}\right)}{2\,\pi\,L\,k_1}$$

$$\dot{Q} = \frac{T_{\infty,1} - T_{\infty,2}}{R_{total}} \qquad R_{total} = R_{conv,1} + R_{cyl,1} + R_{cyl,2} + R_{cyl,3} + R_{conv,2}$$

$$R_{conv,1} = \frac{1}{2\,\pi\,r_1\,L\,h_1} \qquad R_{conv,2} = \frac{1}{2\,\pi\,r_4\,L\,h_2}$$

$$R_{cyl,1} = \frac{\ln\left(\frac{r_2}{r_1}\right)}{2\,\pi\,L\,k_1} \qquad R_{cyl,2} = \frac{\ln\left(\frac{r_3}{r_2}\right)}{2\,\pi\,L\,k_2} \qquad R_{cyl,3} = \frac{\ln\left(\frac{r_4}{r_3}\right)}{2\,\pi\,L\,k_3}$$

The overall heat transfer coefficient, U, can be defined in two ways:

$$U_1 = \frac{1}{A_1\,R_{total}} = \frac{1}{2\,\pi\,r_1\,L\,R_{total}}$$ based on interior surface area

$$U_2 = \frac{1}{A_4\,R_{total}} = \frac{1}{2\,\pi\,r_4\,L\,R_{total}}$$ based on outer surface area

For most engineering problems, the overall heat transfer coefficient, U, is based on the external (outer) surface area because the outer diameter can be easily measured.

$$U_2 = \frac{1}{A_4\ R_{total}} = \frac{1}{2\ \pi\ r_4\ L\ R_{total}}$$

$$U_2 = \frac{1}{\dfrac{r_4}{r_1}\dfrac{1}{h_1} + \dfrac{r_4}{k_1}\ln\left(\dfrac{r_2}{r_1}\right) + \dfrac{r_4}{k_2}\ln\left(\dfrac{r_3}{r_2}\right) + \dfrac{r_4}{k_3}\ln\left(\dfrac{r_4}{r_3}\right) + \dfrac{1}{h_2}}$$

Example 11

A steel tube of 5 cm ID (inner diameter) and 7 cm OD (outer diameter) is covered with 2.5 cm layer of insulation. The inside surface of the tube receives heat by convection from a hot gas while the outer surface of the insulation is exposed to the ambient air. Determine

a) The heat loss to the ambient air per 3 m length of the tube; and

b) The temperature drops across the tube material and the insulation layer.

Data: $T_{\infty,1} = 300\ ^\circ C$ $h_1 = 284\ W/m^2.K$ Neglect radiation.

$T_{\infty,2} = 30\ ^\circ C$ $h_2 = 17\ W/m^2.K$

(a) Radial heat flow through the tube:

$$\dot{Q} = \frac{T_{\infty,1} - T_{\infty,2}}{R_{total}}$$

$$R_{total} = R_{conv,1} + R_{cyl,1} + R_{cyl,2} + R_{conv,2}$$

$$R_{conv,1} = \frac{1}{2\,\pi\, r_1\, L\, h_1} = \frac{1}{2\,\pi\,(0.025)\,(3)\,(284)} = 7.47\ 10^{-3}\ ^{\circ}C\,/\,W$$

$$R_{cyl,1} = \frac{\ln\left(r_2/r_1\right)}{2\,\pi\, L\, k_1} = \frac{\ln\left(3.5/2.5\right)}{2\,\pi\,(3)\,(43.26)} = 4.13\ 10^{-4}\ ^{\circ}C\,/\,W$$

$$R_{cyl,2} = \frac{\ln\left(r_3/r_2\right)}{2\,\pi\, L\, k_2} = \frac{\ln\left((3.5+2.5)/3.5\right)}{2\,\pi\,(3)\,(0.208)} = 1.375\ 10^{-1}\ ^{\circ}C\,/\,W$$

$$R_{conv,2} = \frac{1}{2\,\pi\, r_2\, L\, h_2} = \frac{1}{2\,\pi\,(0.035+0.025)\,(3)\,(17)} = 5.35\ 10^{-2}\ ^{\circ}C\,/\,W$$

$$R_{total} = R_{conv,1} + R_{cyl,1} + R_{cyl,2} + R_{conv,2} = 0.197\ ^{\circ}C / W$$

$$\dot{Q} = \frac{T_{\infty,1} - T_{\infty,2}}{R_{total}} = \frac{300 - 30}{0.197} = 1368\ W$$

(b) Temperature drops

$$\Delta T_{tube} = \frac{R_{cyc,1}}{R_{total}} \left(T_{\infty,1} - T_{\infty,2}\right) = \frac{4.13\ 10^{-4}}{0.197} \left(300 - 30\right) = 0.564\ ^{\circ}C$$

$$\Delta T_{ins} = \frac{R_{cyc,2}}{R_{total}} \left(T_{\infty,1} - T_{\infty,2}\right) = \frac{1.375\ 10^{-1}}{0.197} \left(300 - 30\right) = 188\ ^{\circ}C$$

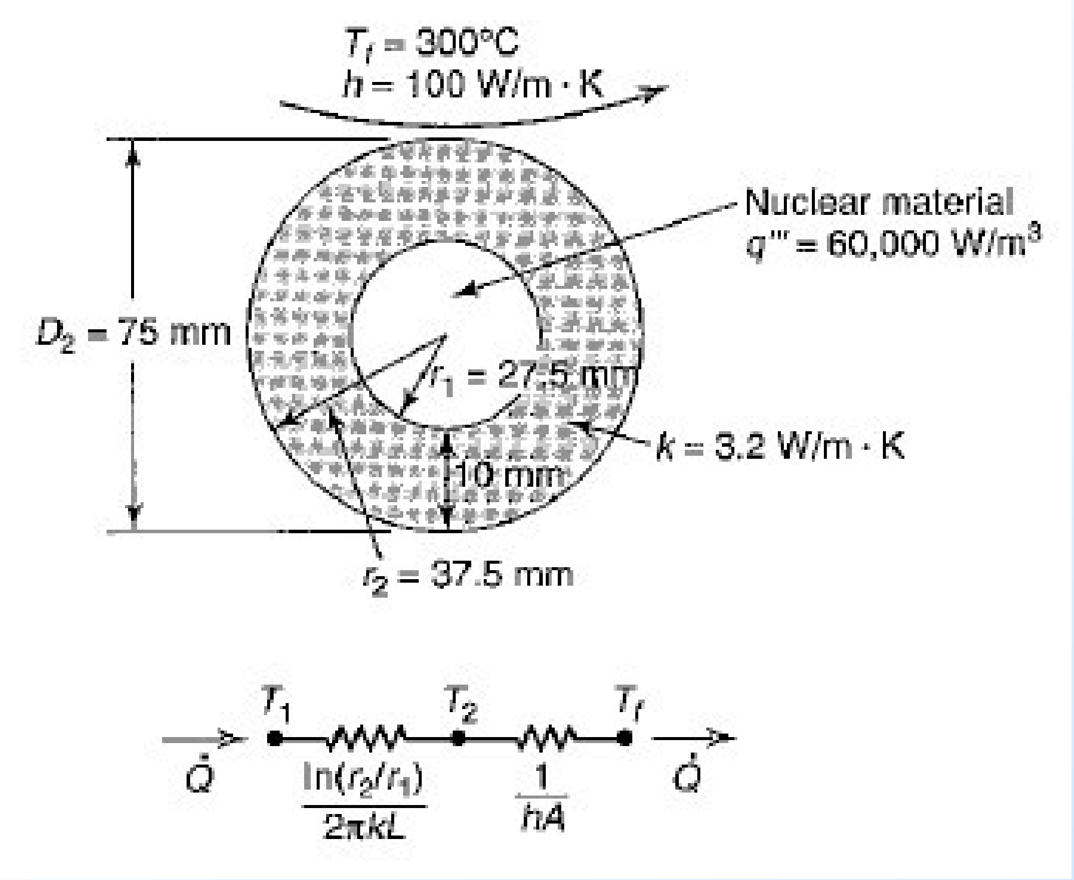
Note the difference in temperature drop

Example 12

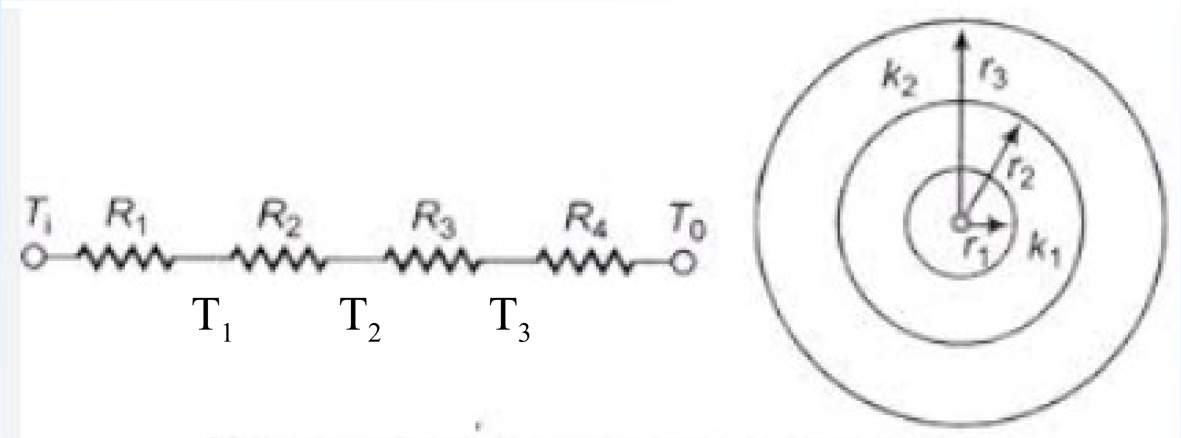
A nuclear fuel rod assembly, consists of an outer cladding and the inner nuclear material, as shown in the figure.

(a) Determine the temperature at the assembly surface (in °C)

(b) Determine the temperature at the interface between the inner nuclear material and the outer cladding (in °C)



3.5.4 Cocentric Spheres



$$\dot{Q} = \frac{T_i - T_o}{R_{total}} = \frac{T_i - T_1}{R_1} = \frac{T_1 - T_2}{R_2} = \frac{T_2 - T_3}{R_3} = \frac{T_3 - T_o}{R_4}$$

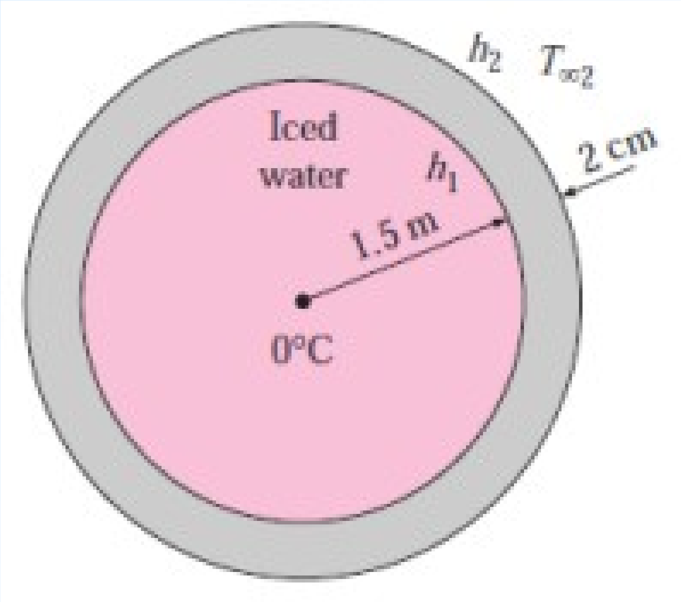
$$R_1 = \frac{1}{4 \pi r_1^2 h_i} \qquad R_2 = \frac{1}{4 \pi k_1} \frac{r_{r2} - r_1}{r_2 \ r_1} \qquad R_3 = \frac{1}{4 \pi k_2} \frac{r_3 - r_2}{r_3 \ r_2} \qquad R_4 = \frac{1}{4 \pi r_3^2 h_0}$$

For most engineering problems, the overall heat transfer coefficient, U, is based on the external (outer) surface area because the outer diameter can be easily measured.

$$U_0 = \frac{1}{A_o\ R_{total}} = \frac{1}{4\ \pi\ r_3^2\ R_{total}}$$

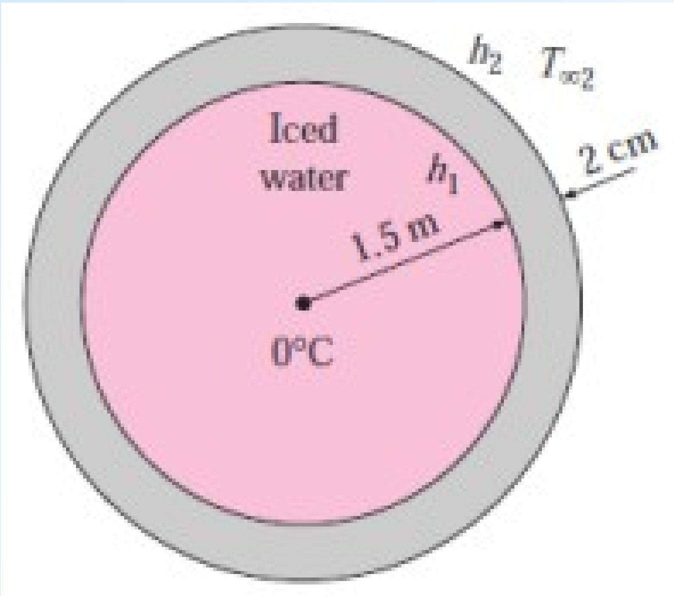
$$U_o = \frac{1}{\frac{1}{h_1}\frac{r_3^2}{r_1^2} + \frac{r_2 - r_1}{k_1}\frac{r_3^2}{r_2\ r_1} + \frac{r_3 - r_2}{k_2}\frac{r_3^2}{r_3\ r_2} + \frac{1}{h_0}}$$

Example 13

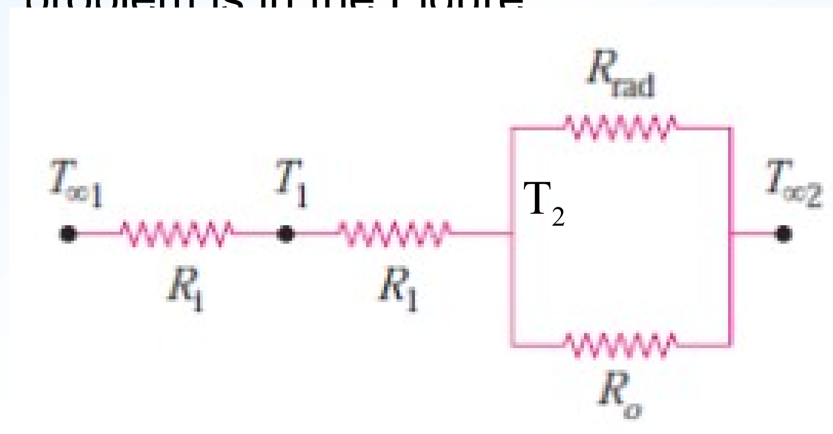


A 17 m internal diameter spherical tank made of 2 cm thick stainless steel (k = 15 W/m.K) is used to store iced water at $T_{\infty,1}$ = 0 ºC. the tank is located in a room whose temperature is $T_{\infty,2}$ = 22 ºC. The walls of the room are also at 22 ºC. The outer surface of the tank is black and heat transfer between the outer surface of the tank and the surroundings is by natural convection and radiation.

The convection heat transfer coefficients at the inner and the outer surfaces of the tank are h_1 = 80 W/m^2.K and h_2 = 10 W/m^2.K, respectively. Determine the rate of heat transfer to the iced water in the tank.



The thermal resistance network for this problem is in the Figure



Inner surface area: $A_1 = \pi D^2{}_1 = 28.3\ m^2$

Inner diameter: $D_1 = 3$ m

Outer surface area: $A_2 = \pi D^2{}_2 = 29\ m^{2c}$

Outer diameter $D_2 = 3.04$ m

The radiation heat transfer coefficient is given by: $h_r = \varepsilon\ \sigma\ (T_2^2 + T_{\infty,2}^2)\ (T_2 + T_{\infty,2})$

T_2 is unkown. In order to calculate, h_r, a trial-and–error procedure is necessary.

Assume value for T_2, check this assumption later, and repeat the calculations if necessary using a revised value of T_2.

Note that T_2, must be between 0 ºC and 22 ºC, and closer to 0 ºC since the heat transfer coefficient inside the tank is much larger.

Take T_2 = 5 ºC = 278 K. Then

$$h_r = \varepsilon\,\sigma\,(T_2^2 + T_{\infty,2}^2)\,(T_2 + T_{\infty,2}) = (1)\,(5.67\;10^{-8})\,\left((295)^2 + (278)^2\right)\,\left(295 + 278\right)$$

$$= 5.34\ \text{W/m}^2.\text{K}$$

The other thermal resistances are:

$$R_i = R_{conv,1} = \frac{1}{h_1\,A_1} = \frac{1}{(80)\,(28.3)} = 0.000442\ ^{\circ}\text{C/W}$$

$$R_1 = R_{sph} = \frac{r_2 - r_1}{4\,\pi\,k\,r_1\,r_2} = \frac{1.52 - 1.50}{4\,\pi\,(15)\,(1.52)\,(1.50)} = 0.000047\ ^{\circ}\text{C/W}$$

$$R_0 = R_{conv,2} = \frac{1}{h_2 \; A_2} = \frac{1}{(10)\,(29)} = 0.00345 \; ^\circ C/W$$

$$R_{rad} = \frac{1}{h_r \; A_2} = \frac{1}{(5.34)\,(29)} = 0.00646 \; ^\circ C/W$$

Parallel resistances, R_0 and R_{rad}, can be replaced by an equivalent resistance:

$$\frac{1}{R_{eq}} = \frac{1}{R_0} + \frac{1}{R_{rad}} = \frac{1}{0.00345} + \frac{1}{0.00646} = 444.7 \; W/^\circ C \qquad R_{eq} = 0.00225 \; ^\circ C/W$$

$$R_{tot} = R_i + R_1 + R_{eq} = 0.000442 + 0.000047 + 0.0025 = 0.00273 \; ^\circ C/W$$

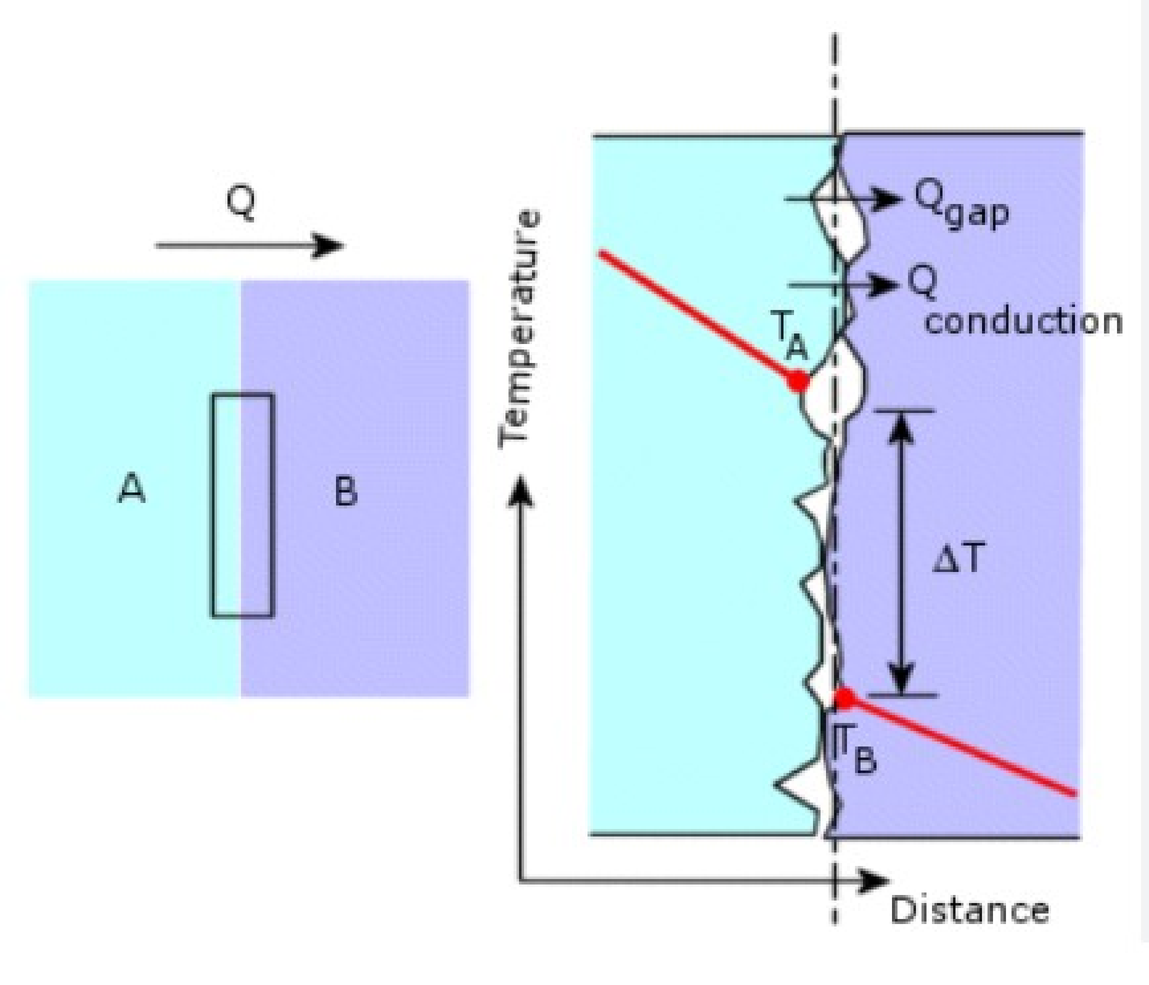
$$\dot{Q} = \frac{T_{\infty,2} - T_{\infty,1}}{R_{tot}} = \frac{22 - 0}{0.00274} = 8029 \; W$$

To check the validity of our original assumption (T_2 = 5 ºC), determine the outer surface temperature from

$$\dot{Q} = \frac{T_{\infty,2} - T_2}{R_{eq}} \quad \Rightarrow \quad T_2 = T_{\infty,2} - \dot{Q}\, R_{eq} = 22 - (8029)\,(0.00225) = 4\ ^{\circ}\mathrm{C}$$

The calculations need not to be repeated with T_2 = 4 ºC. Why?

3.6 Thermal Contact Resistance



If two solids are not metallurgically bonded together, the anlarged view is as shown in the Figure.

The heat transfer across the actual contact points and the small air (gas) gaps is mainly by conduction. Radiation is negligible at room temperatures.

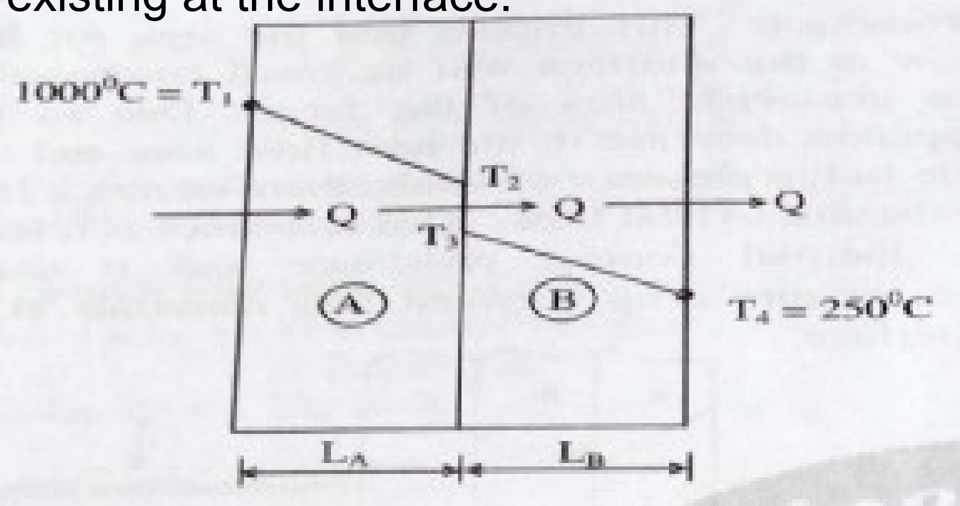
The thermal conductivity of a gas (air) is smaller than that of a solid. Therefore, the rate of heat flow is reduced due to the presence of stagnant gas in the gaps. The extra resistance to heat flow is called thermal contact resistance.

The interface thermal conductance h in $W/m^2.K$ is determined by experiments. The reciprocal of h, 1/h, is called specific thermal contact resistance. R_{cont}.

h increases with increasing interface pressure (increased solid to solid contact area) and decreases with increasing surface roughness and waviness.

Example 14

A boiler wall is made up of two layers, A and B. Thickness and thermal conductivity of A are L_A = 240 mm and k_A = 0.2 W/m.°C, respectively. For B, thickness and thermal conductivity are L_B = 525 mm and k_B = 0.3 W/m.°C, respectively. Inner surface of A is maintained at T_1 = 1000 °C and outer surface of B is maintained at T_4 = 250 °C. There is a contact thermal resistance of R_{cont} = 0.050 °C/W per unit area existing at the interface.



Calculate

(a) The heat lost per m² area;

(b) The temperature drop at the interface.

Heat flux: $$\dot{q} = \frac{T_1 - T_2}{\frac{L_A}{k_A}} = \frac{T_3 - T_4}{\frac{L_B}{k_B}} = \frac{T_1 - T_4}{\frac{L_A}{k_A} + R_{cont} + \frac{L_B}{k_B}}$$

$$\dot{q} = \frac{T_1 - T_4}{\frac{L_A}{k_A} + R_{cont} + \frac{L_B}{k_B}} = \frac{1000 - 250}{\frac{0.24}{0.2} + 0.05 + \frac{0.525}{0.3}} = 250 \text{ W/m}^2$$

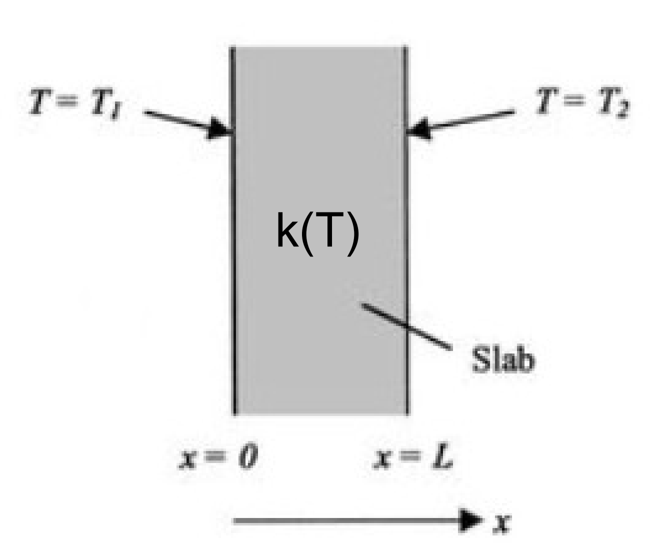
$$250 = \frac{T_1 - T_2}{\frac{L_A}{k_A}} = \frac{1000 - T_2}{\frac{0.24}{0.2}} \quad \Rightarrow \quad T_2 = 700\ ^\circ\text{C}$$

$$250 = \frac{T_3 - T_4}{\frac{L_B}{k_B}} = \frac{T_3 - 250}{\frac{0.525}{0.3}} \quad \Rightarrow \quad T_3 = 687.5\ ^\circ\text{C}$$

3.7 Variable Thermal Conductivity

If we have a solid whose thermal conductivity strongly varies with temperature or if temperature differences are quite large, then this dependence has to be accounted for. In general, the solutions are complicated, but for one dimensional, steady state case, they are straight forward and relatively easier.

3.7.1 Slab with Variable Thermal Conductivity



Differential Equation: $$\frac{d}{dx}\left(k(T)\, \frac{dT}{dx} \right) = 0$$

Boundary Conditions:
(1) $T = T_0$ at $x = 0$
(2) $T = T_1$ at $x = L$

Question: $\dot{Q} = ?$

Solution: $k(T)\dfrac{dT}{dx} = C \quad \Rightarrow \quad k(T)\,dT = C\,dx$

If k(T) is a known function of temperature, the distribution, T(x), can easily be found.

Fourier's law: $Q = A\,\dot{q} = -A\,k(T)\dfrac{dT}{dx} = -A\,C$ constant

Integrate both sides to find C: $\displaystyle\int_{T_0}^{T_1} k(T)\,dT = C\int_0^L dx = C\,L \quad \Rightarrow \quad C = \frac{-1}{L}\int_{T_0}^{T_1} k(T)\,dT$

If k(T) is a linear function of temperature such as $k(T) = k_0\,(1 + \beta\,T)$

$$Q = \frac{A}{L}\int_{T_0}^{T_1} k_0\,(1 + \beta\,T)\,dT = A\,k_0\left(1 + \beta\,\frac{T_0 + T_1}{2}\right)\left(\frac{T_0 - T_1}{L}\right)$$

If mean thermal conductivity is defined as $$k_m = k_0 \left(1 + \beta \frac{T_0 + T_1}{2}\right)$$

$$Q = A\, k_m \left(\frac{T_0 - T_1}{L}\right)$$

This is the same solution as before except that k is evaluated at the arithmetic mean temperature.

Example 15

Determine the heat flux across 15 cm thick slab when one face is kept at T_1 = 500 K an the other face at T_2 = 300 K. The thermal conductivity varies linearly wlth temperature as k(T) = k_0 (1 + β T) where k_0 = 0.0346 W/m.K and β = 0.0036 K^{-1}.

$$\dot{q} = k_0 \left(1 + \beta \, \frac{T_1 + T_2}{2}\right)\left(\frac{T_1 - T_2}{L}\right)$$

$$= (0.0346) \left[1 + (0.0036 \left(\frac{500 + 300}{2}\right)\right]\left(\frac{500 - 300}{0.15}\right)$$

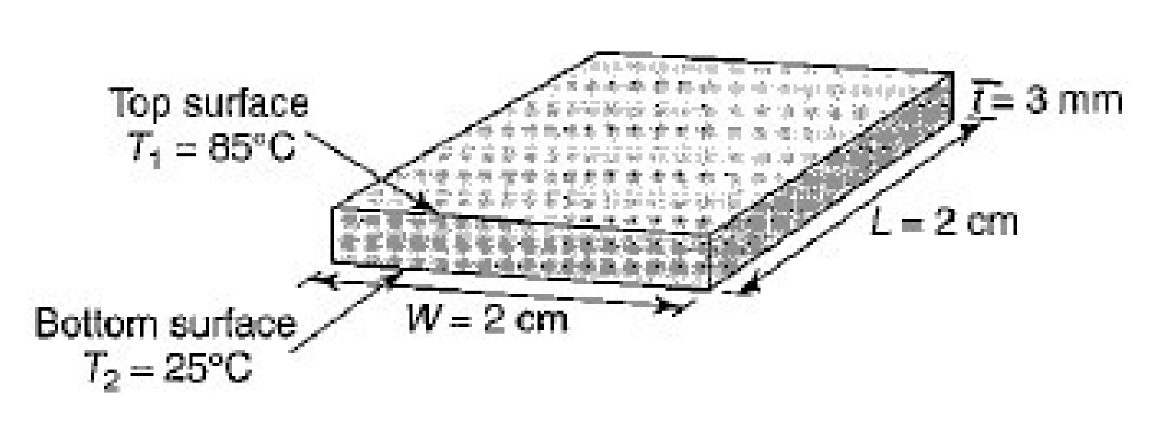
$$= 112.57 \text{ W/m}^2$$

Example 16

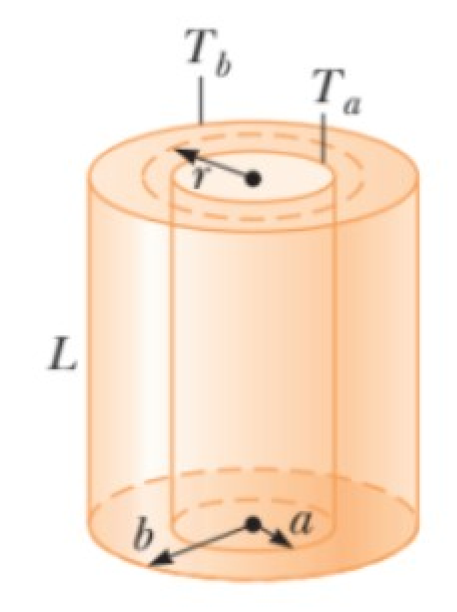
A wafer of silicon, 3 mm thick and 2 cm square, is used in an electronic device. One side of the device is held at 85 °C and the other is held at 25 °C. The thermal conductivity of silicon varies with temperature as $k = k_0 (1 + \beta T)$, where $k_0 = 15$ W/m.K, $\beta = 0.00556$ °C^{-1}, and T is in °C.

(a) Determine the HT rate (in W) if k is evaluated at its average temperature.

(b) Determine the HT rate (in W) if the temperature dependence of k is taken into account.



3.7.2 Hollow Cylinder with Variable Thermal Conductivity



Differential Equation: $$\frac{d}{dr}\left(r\,k(T)\,\frac{dT}{dr}\right) = 0$$

Boundary Conditions:
(1) $T = T_a$ at $r = a$
(2) $T = T_b$ at $r = b$

Question: $\dot{Q} = ?$

Solution: $$r\,k(T)\,\frac{dT}{dr} = C \quad \Rightarrow \quad k(T)\,dT = C\,\frac{dr}{r} \quad \Rightarrow \quad \int_{T_a}^{T_b} k(T)\,dT = C\int_a^b \frac{dr}{r}$$

$$C = \frac{-1}{\ln\left(b/a\right)}\int_{T_a}^{T_b} k(T)\,dT$$

$$\dot{Q} = 2\,\pi\, r\, L\, \dot{q} = -2\,\pi\, r\, L \left[k(T)\, \frac{dT}{dr} \right] = -2\,\pi\, r\, L\, \frac{C}{r} = -2\,\pi\, L\, C$$

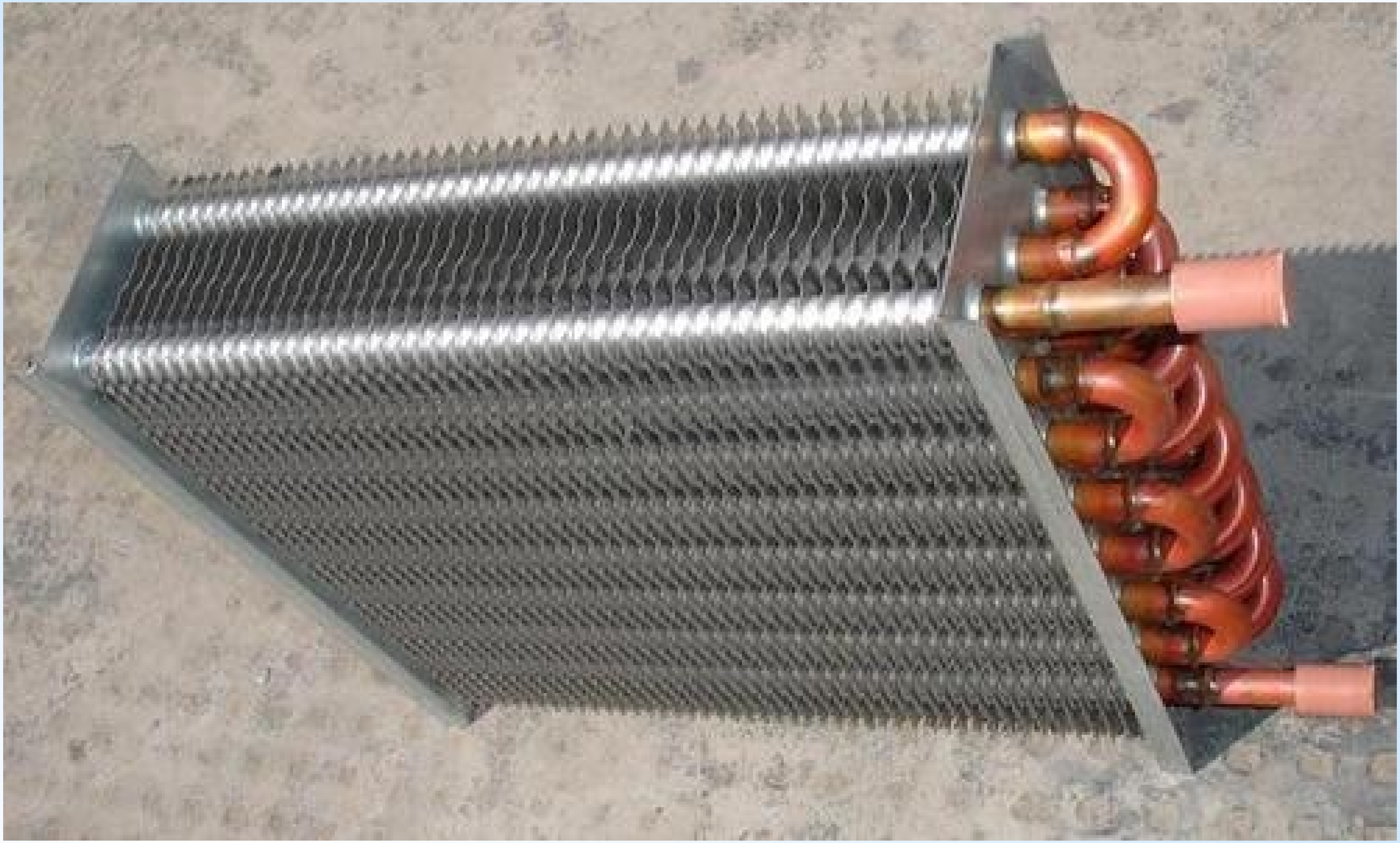
$$\dot{Q} = \frac{2\,\pi\, L}{\ln\left(b/a\right)} \int_{T_a}^{T_b} k(T)\, dT$$

If k(T) is known, Q is calculated, easily.

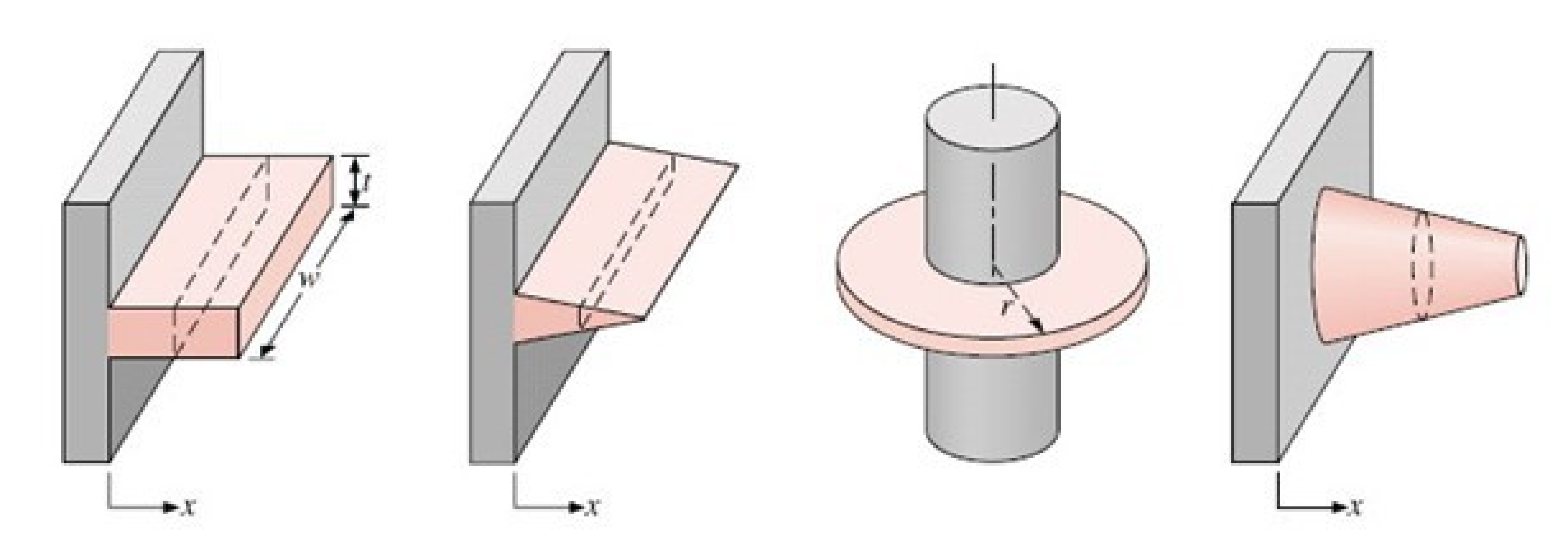
3.8 One-dimensional Fin Equation

Thins trips of metals called **fins (or extended surfaces)** are attached to the surface of a solid to inrease the heat transfer area and hence the heat transfer by convection between the surface and the fluid surrounding it.

Fins are generally used on the surface where the convective heat transfer coefficient is low. Example: car radiator.



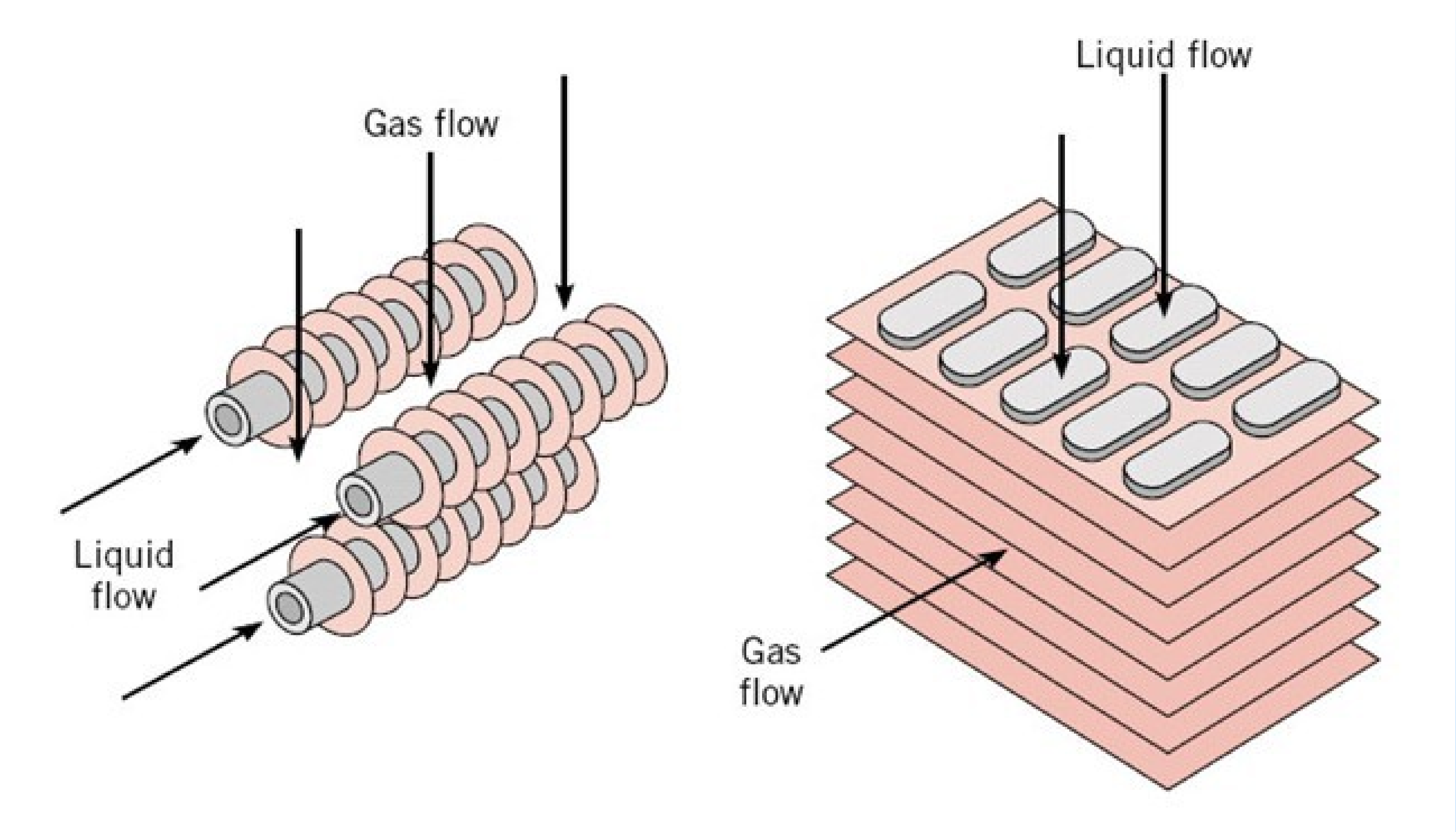
Examples of Extended Surfaces - Fins

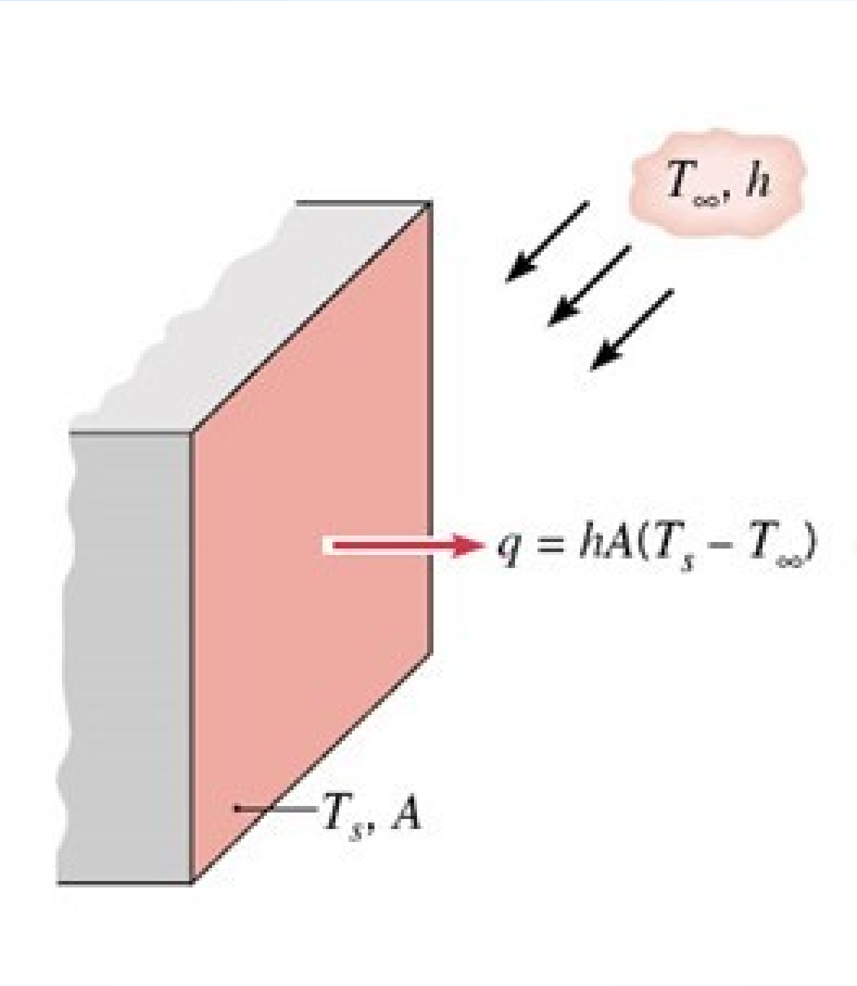


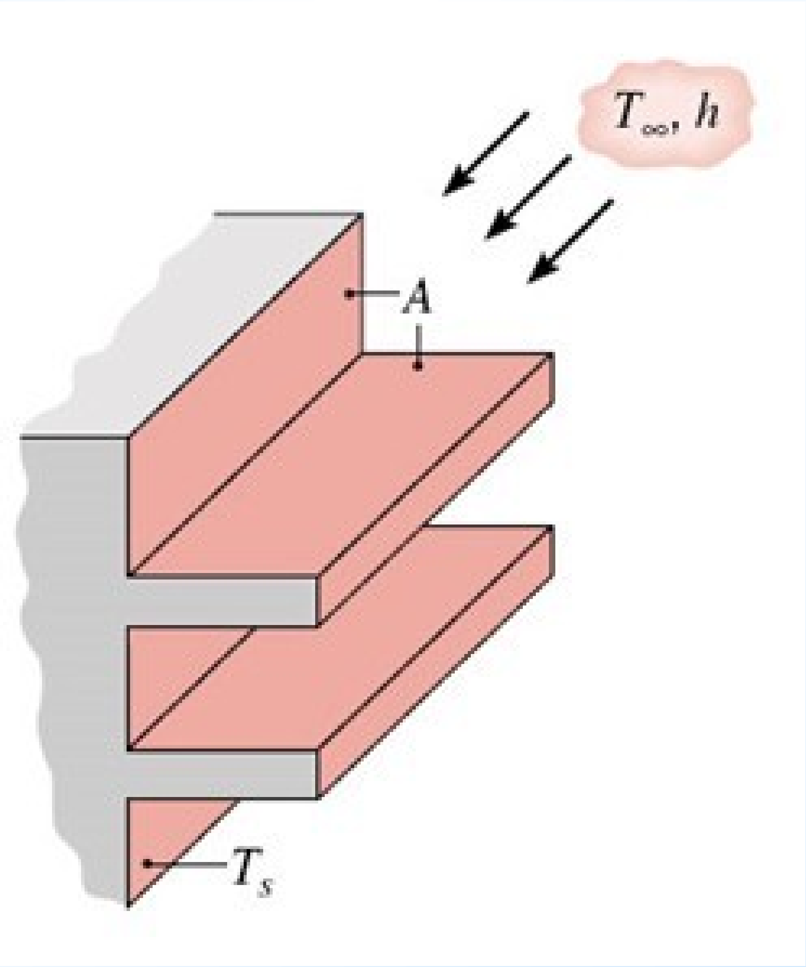
Straight Fins
uniform and non-uniform cross section

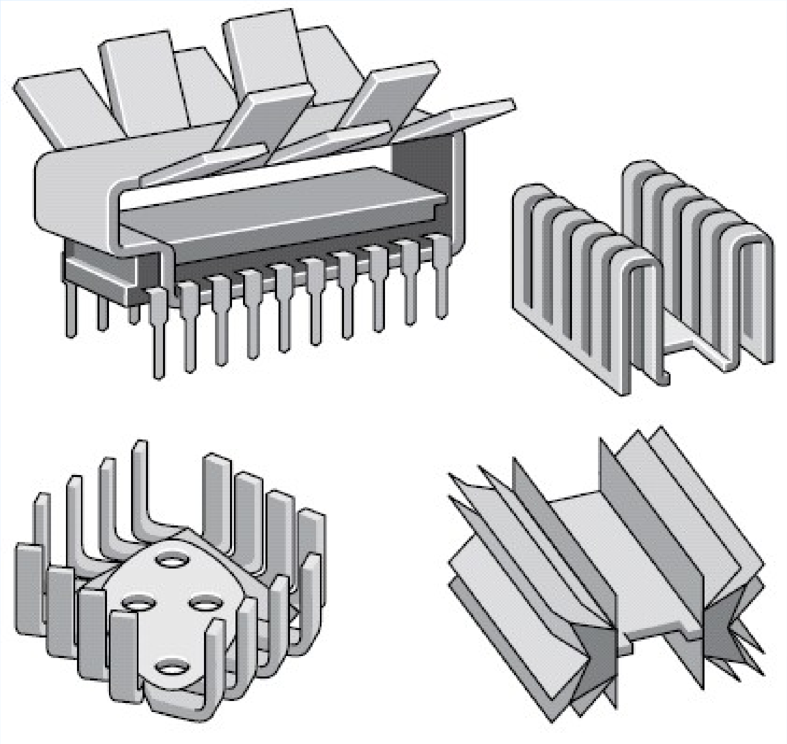
Annular Fin

Pin Fin


Gas flow
Liquid flow
Liquid flow
Gas flow


T_∞, h
$q = hA(T_s - T_\infty)$
T_s, A


T_∞, h
A
T_s



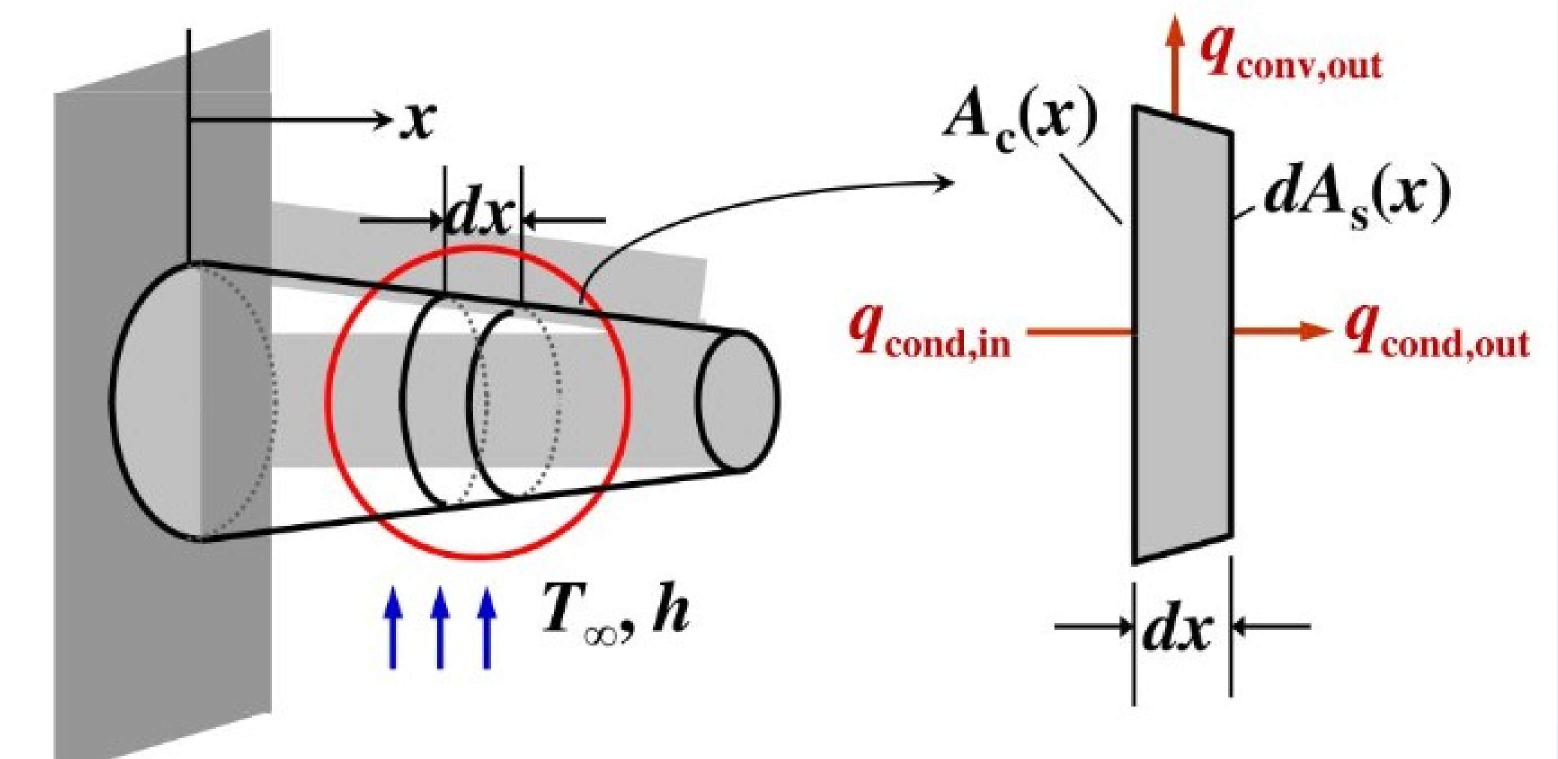
Newton's Law of Cooling:

$$\dot{Q}_{conv} = A\,h\,(T_s - T_\infty)$$

Two ways to increase the rate of heat transfer:

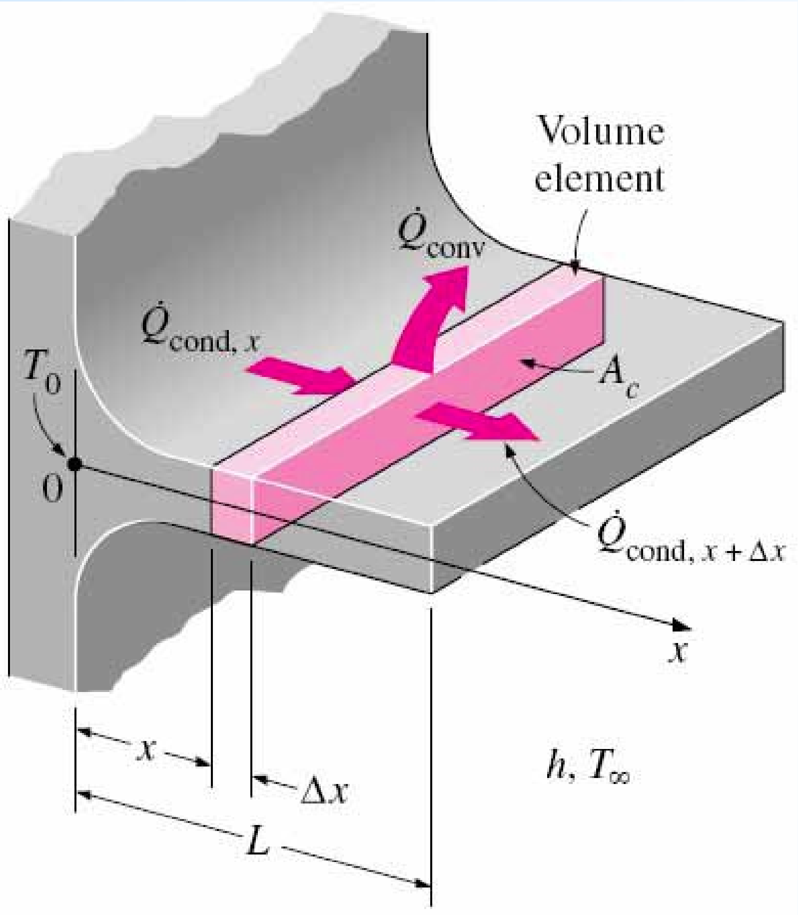
- increasing the heat transfer coefficient;
- increase the surface area

⇨ fins



$$q_{cond,in} = q_{cond,out} + q_{conv,out}$$

3.9 Temperature Distribution and Heat Flow in Fins of Uniform Cross Section



Under steady-state conditions, the energy balance on this volume can be expressed as

$$\begin{pmatrix}\text{Rate of heat}\\ \text{conduction into}\\ \text{the element at x}\end{pmatrix} = \begin{pmatrix}\text{Rate of heat}\\ \text{conduction from}\\ \text{the element at x+}\Delta\text{x}\end{pmatrix} + \begin{pmatrix}\text{Rate of heat}\\ \text{convection from}\\ \text{the element}\end{pmatrix}$$

$$\dot{Q}_{cond,x} = \dot{Q}_{cond,x+\Delta x} + \dot{Q}_{conv}$$

$$\dot{Q}_{cond,x} = \dot{Q}_{cond,x+\Delta x} + \dot{Q}_{conv}$$

$$= \dot{Q}_{cond,x+\Delta x} + h\,(p\,\Delta x)\,(T - T_\infty)$$ where p is the perimeter

$$\frac{\dot{Q}_{cond,x+\Delta x} - \dot{Q}_{cond,x}}{\Delta x} = h\,p\,(T - T_\infty)$$

As Δx goes to zero: $$\frac{d\dot{Q}_{cond}}{dx} = h\,p\,(T - T_\infty)$$

From Fouries' Law: $$\dot{Q}_{cond} = -k\,A_c\,\frac{dT}{dx}$$ where A_c is the cross-sectional area

$$\frac{d}{dx}\left(k\,A_c\,\frac{dT}{dx}\right) - h\,p\,(T - T_\infty) = 0$$

$$\frac{d}{dx}\left(k A_c \frac{d\,T(x)}{dx}\right) - h\,p\,\left(T(x) - T_\infty\right) = 0$$

For constant cross section, A_c, and thermal conductivity, k, and

defining: $\theta(x) = T(x) - T_\infty$ and $m = \sqrt{\dfrac{h\,p}{k A_c}}$

$$\frac{d^2\theta}{dx^2} - m^2\,\theta = 0$$

General Solution: $\theta(x) = C_1\,e^{m x} + C_2\,e^{-m x}$

The constants, C_1 and C_2, are to be determined from boundary conditions

Fin Equation: $$\frac{d^2\theta}{dx^2} - m^2\,\theta = 0$$ $\theta(x) = T(x) - T_\infty$ and $m = \sqrt{\frac{h\,p}{k\,A_c}}$

General Solution(s):

$$\theta(x) = C_1\,e^{mx} + C_2\,e^{-mx}$$

$$\theta(x) = C_3\,\cosh(m\,x) + C_4\,\sinh(m\,x)$$

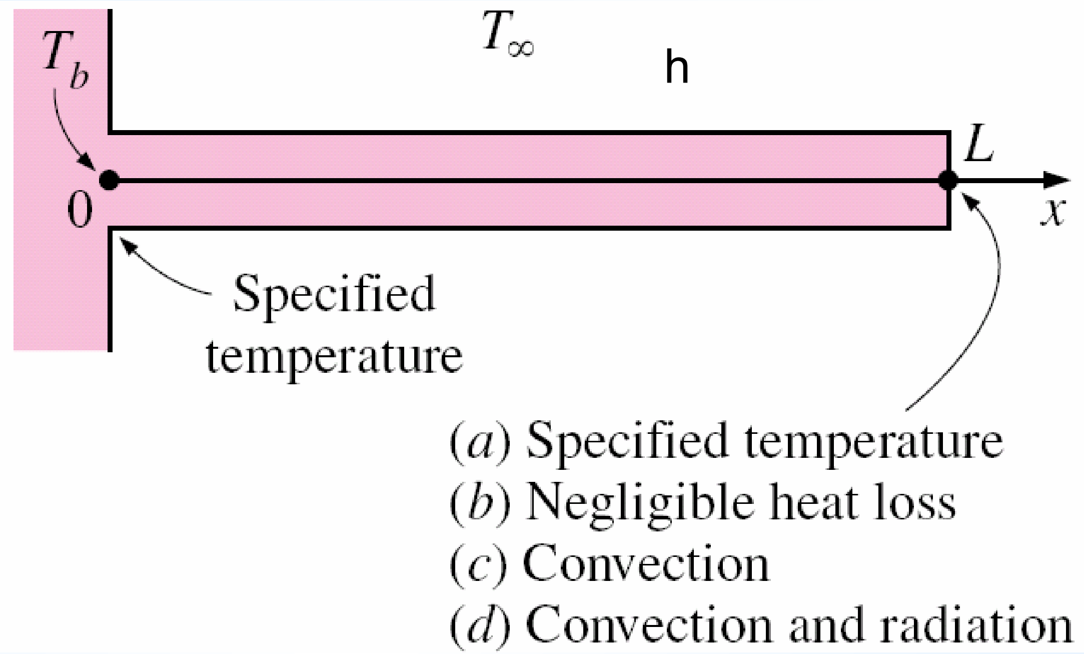
$$\theta(x) = C_5\,\cosh\big(m\,(L-x)\big) + C_6\,\sinh\big(m\,(L-x)\big)$$

Which general solution is used depends on the given boundary conditions. Use the one which is easier to apply the BC's to determine the constants.

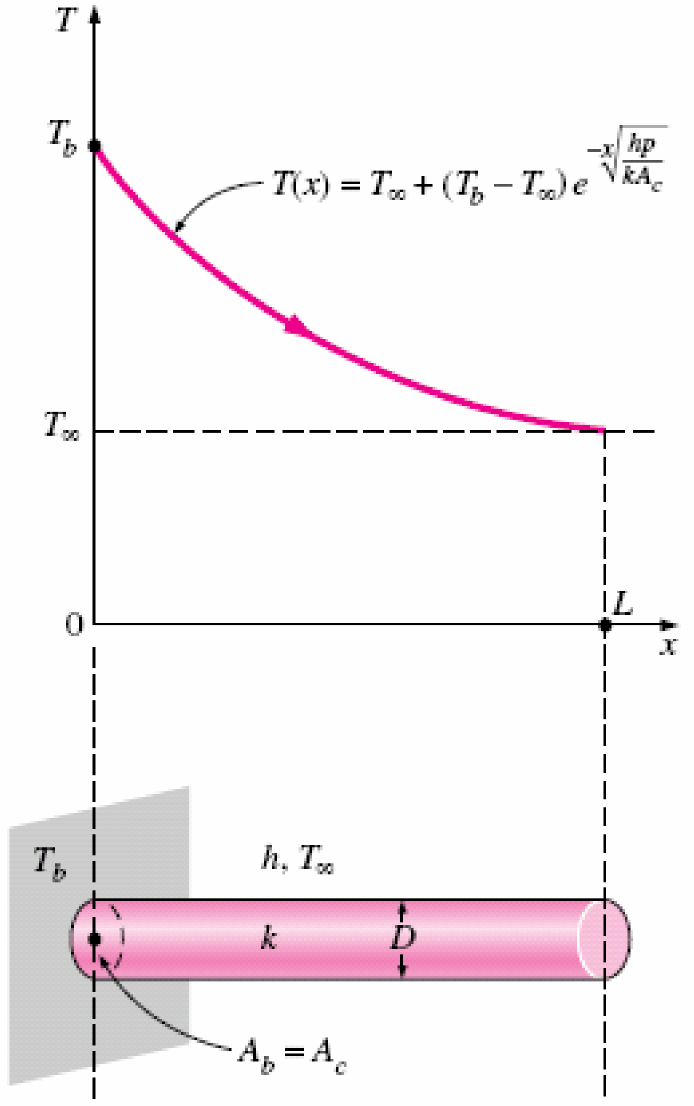
Remember the definitions to be used to convert one solution to the other.

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \quad \text{and} \quad \sinh(x) = \frac{e^x - e^{-x}}{2}$$

Boundary Conditions



At the fin base (x = 0): $\theta(0) = \theta_b = T_b - T_\infty$



3.9.1 Infinitely(!) Long Fin $T_{fin\ tip} \Rightarrow T_\infty$

For a sufficiently long fin, the temperature at the fin tip approaches the ambient temperature, i.e., T(L) => T_∞

The boundary condition becomes

$$\theta(x \rightarrow \infty) = T(L) - T_\infty \cong 0$$

Question: How long is "long"?

$$L \geq 5\sqrt{\frac{k\,A_c}{h\,p}}$$

Infinitely(!) Long Fin $T_{\text{fin tip}} \Rightarrow T_\infty$

Fin Equation: $\dfrac{d^2\theta}{d\,x^2} - m^2\,\theta = 0 \qquad m = \sqrt{\dfrac{h\,p}{k\,A_c}}$

General Solution: $\theta(x) = C_1\,e^{m\,x} + C_2\,e^{-m\,x}$

Boundary Conditions: $\theta(x \to \infty) = T(L) - T_\infty = 0 \quad \Rightarrow \quad C_2 \equiv 0$

$$\theta(0) = \theta_b = T_b - T_\infty \quad \Rightarrow \quad C_1 = \theta_b = T_b - T_\infty$$

$$\theta(x) = \theta_b\,e^{-m\,x} \quad \Rightarrow \quad \frac{T(x) - T_\infty}{T_b - T_\infty} = e^{-m\,x} = e^{-x\sqrt{\frac{h\,p}{k\,A_c}}}$$

Infinitely(!) Long Fin $T_{\text{fin tip}} \Rightarrow T_\infty$

$$\theta(x) = \theta_b\ e^{-m x} \quad \Rightarrow \quad \frac{T(x) - T_\infty}{T_b - T_\infty} = e^{-m x} = e^{-x\sqrt{\frac{h p}{k A_c}}}$$

Total rate of heat transfer from the fin:

$$\dot{Q} = \int_{x=0}^{x=\infty} h\ p\ \theta(x)\ dx$$

$$\dot{Q} = -A_c\ k \left.\frac{d\theta}{dx}\right|_{x=0}$$

Both equations should give the same result:

$$\dot{Q} = A_c\ k\ \theta_b\ m = \theta_b \sqrt{p\ h\ k\ A_c}$$

$$= (T_b - T_\infty)\sqrt{p\ h\ k\ A_c}$$

3.9.2 Fin with Adiabatic Tip (negligible heat flow at the tip)

Fin Equation: $$\frac{d^2\theta}{d\,x^2} - m^2\,\theta = 0$$

General Solution: $$\theta(x) = C_3\,\cosh\big(m\,(L - x)\big) + C_4\,\sinh\big(m\,(L - x)\big)$$

Boundary Conditions: $$\left.\frac{d\,\theta}{d\,x}\right|_{x=L} = 0 \quad\Rightarrow\quad C_4 \equiv 0$$

$$\theta(0) = \theta_b = T_b - T_\infty \quad\Rightarrow\quad C_3 = \frac{\theta_b}{\cosh(m\,L)}$$

$$\theta(x) = \theta_b\,\frac{\cosh\big(m\,(L - x)\big)}{\cosh(m\,L)}$$

Fin with Adiabatic Tip (negligible heat flow at the tip)

Temperature profile: $$\theta(x) = \theta_b \, \frac{\cosh\left(m\,(L - x)\right)}{\cosh(m\,L)}$$

Total rate of heat transfer from the fin:

$$\dot{Q} = -A\,k\,\left.\frac{d\theta}{dx}\right|_{x=0}$$

$$\dot{Q} = \theta_b\, m\, A_c\, k \tanh(m\,L)$$

$$= (T_b - T_\infty)\sqrt{p\,h\,k\,A_c}\,\tanh(m\,L)$$

3.9.3 Fin with Convection at the Tip

Fin Equation: $$\frac{d^2\theta}{d\,x^2} - m^2\,\theta = 0$$

General Solution: $$\theta(x) = C_3\,\cosh\big(m\,(L - x)\big) + C_4\,\sinh\big(m\,(L - x)\big)$$

Boundary Conditions:

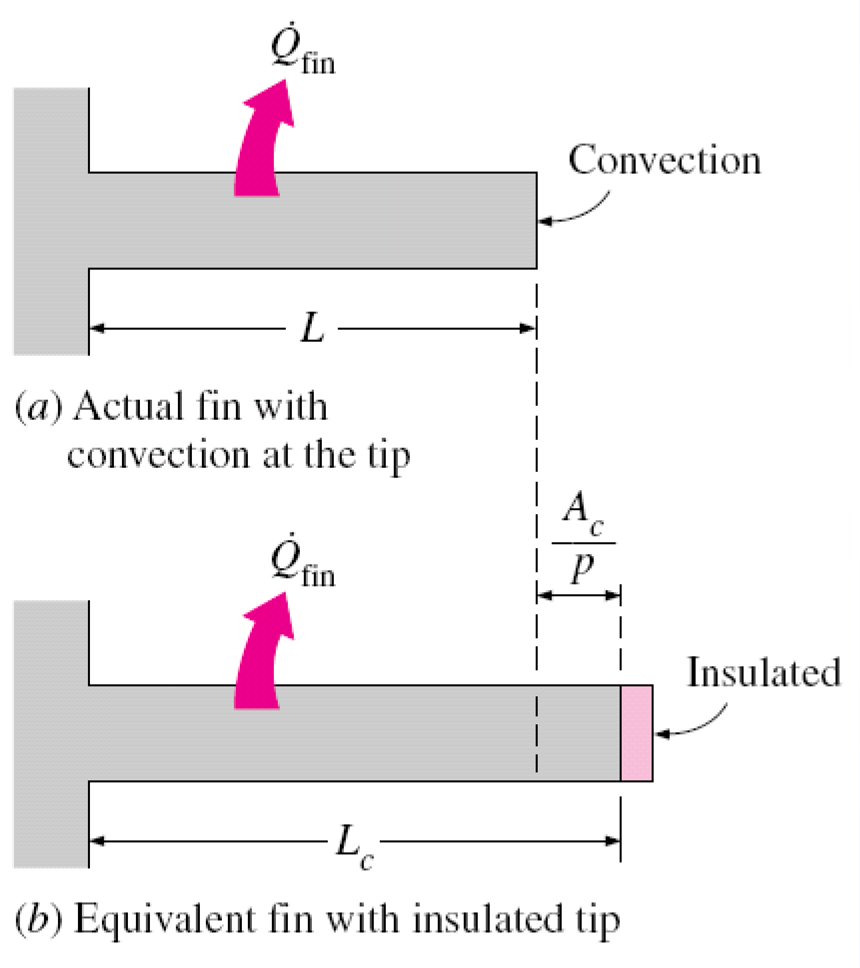
$$k\,\left.\frac{d\,\theta}{d\,x}\right|_{x=L} + h_{tip}\,\theta(L) = 0$$

$$\theta(0) = \theta_b = T_b - T_\infty$$

h_{tip} is not necessarily equal to h

$$\theta(x) = \theta_b\,\frac{\cosh\big(m\,(L - x)\big) + \dfrac{h_{tip}}{m\,k}\,\sinh\big(m\,(L - x)\big)}{\cosh(m\,L) + \dfrac{h_{tip}}{m\,k}\,\sinh(m\,L)}$$

Fin with Convection at the Tip



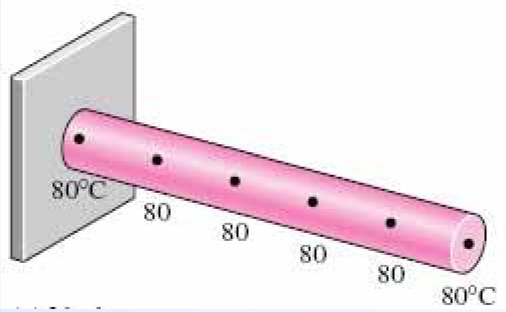
A practical way of accounting for the heat loss from the fin tip is to replace the *fin length L in the relation* for the *insulated tip case by a* ***corrected length*** defined as

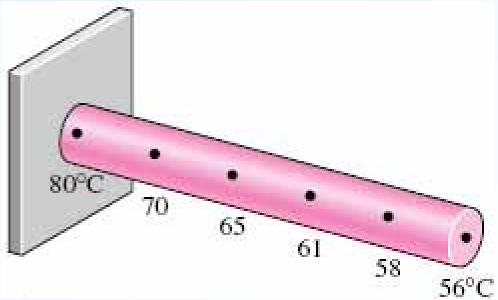
$$L_c = L + \frac{A_c}{p}$$

$$L_{c,rectangular} \cong L + \frac{t}{2}$$

$$L_{c,cylindrical} \cong L + \frac{D}{4}$$

3.10 Fin Efficiency





To maximize the heat transfer from a fin, the temperature of the fin should be uniform (maximized) at the base value of T_b .

In reality, the temperature drops along the fin, and thus the heat transfer from the fin is less.

To account for the effect, we define a **fin efficiency:**

$$\eta_{fin} = \frac{\dot{Q}_{fin}}{\dot{Q}_{fin,\max}} = \frac{\text{Actual Heat Transfer Rate from the Fin}}{\text{Ideal Heat Transfer Rate from the Fin if the entire fin were at the base temperature}}$$

$$\dot{Q}_{fin} = \eta_{fin}\ h\ A_{fin}\ (T_b - T_\infty)$$

In most cases, fins have variable cross sectional areas and that makes the solution complicated. For a variety of fin geometries the calculations are presented in terms the fin efficiency.

$$\dot{Q}_{fin} = \eta_{fin}\, h\, A_{fin}\, (T_b - T_\infty) = \eta_{fin}\, h\, A_{fin}\, \theta_b$$

For the case of a fin with negligible heat flow at the tip:

$A_{fin} = p\,L$ Where p = Perimeter of the fin and L = Length of the fin

$\dot{Q}_{ideal} = p\,L\,h\,\theta_b$ $\dot{Q}_{fin} = \theta_b \sqrt{p\,h\,k\,A_{fin}}\,\tanh(m\,L)$ as found before

$$\eta_{fin} = \frac{\dot{Q}_{fin}}{\dot{Q}_{ideal}} = \frac{\theta_b \sqrt{p\,L\,k\,A_{fin}}\,\tanh(m\,L)}{\theta_b\, p\,L\,h}$$

$$\boxed{\eta_{fin} = \frac{\tanh(m\,L)}{m\,L}}$$

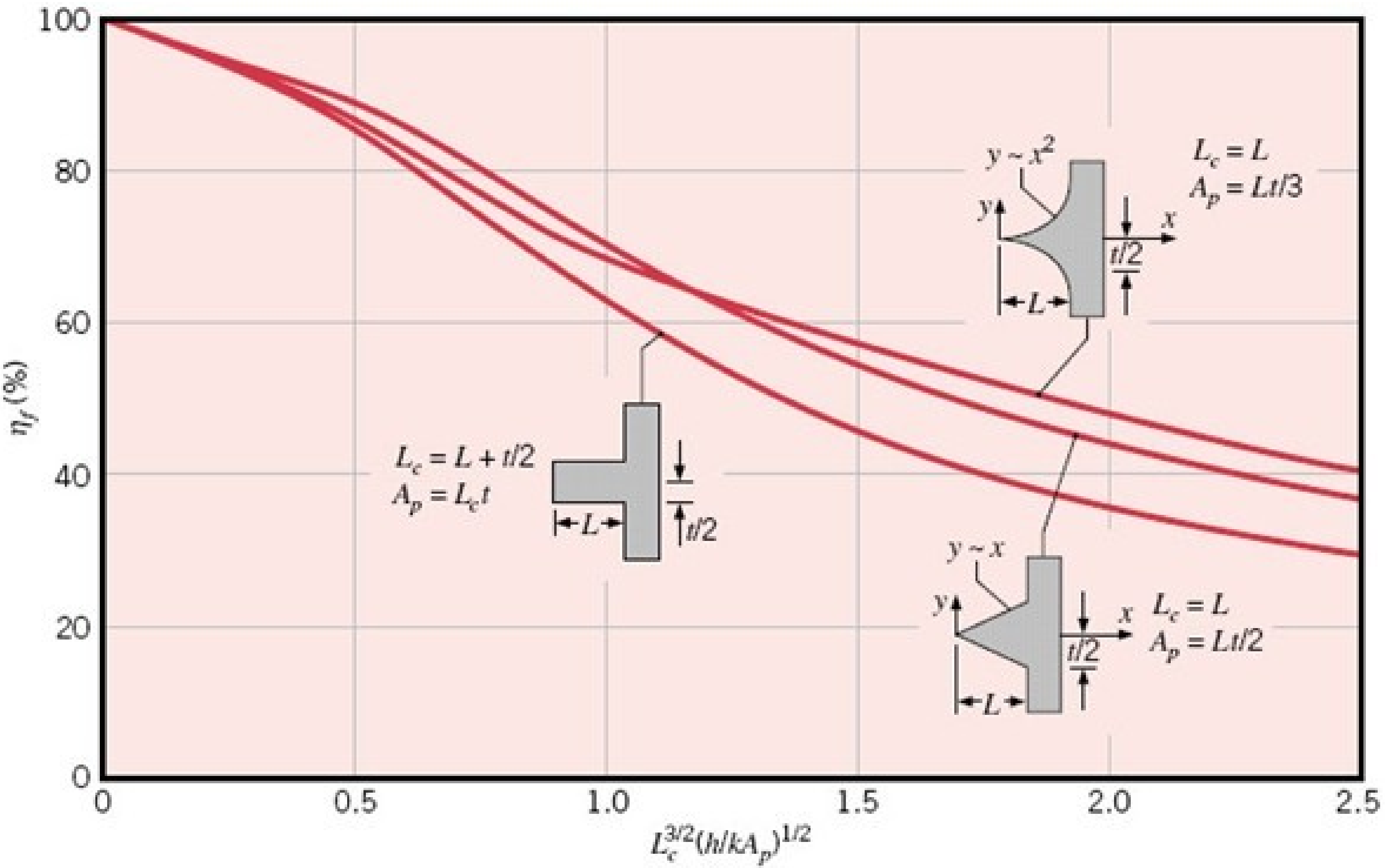
There are charts available that gives η_{fin} for various fin shapes as a function of

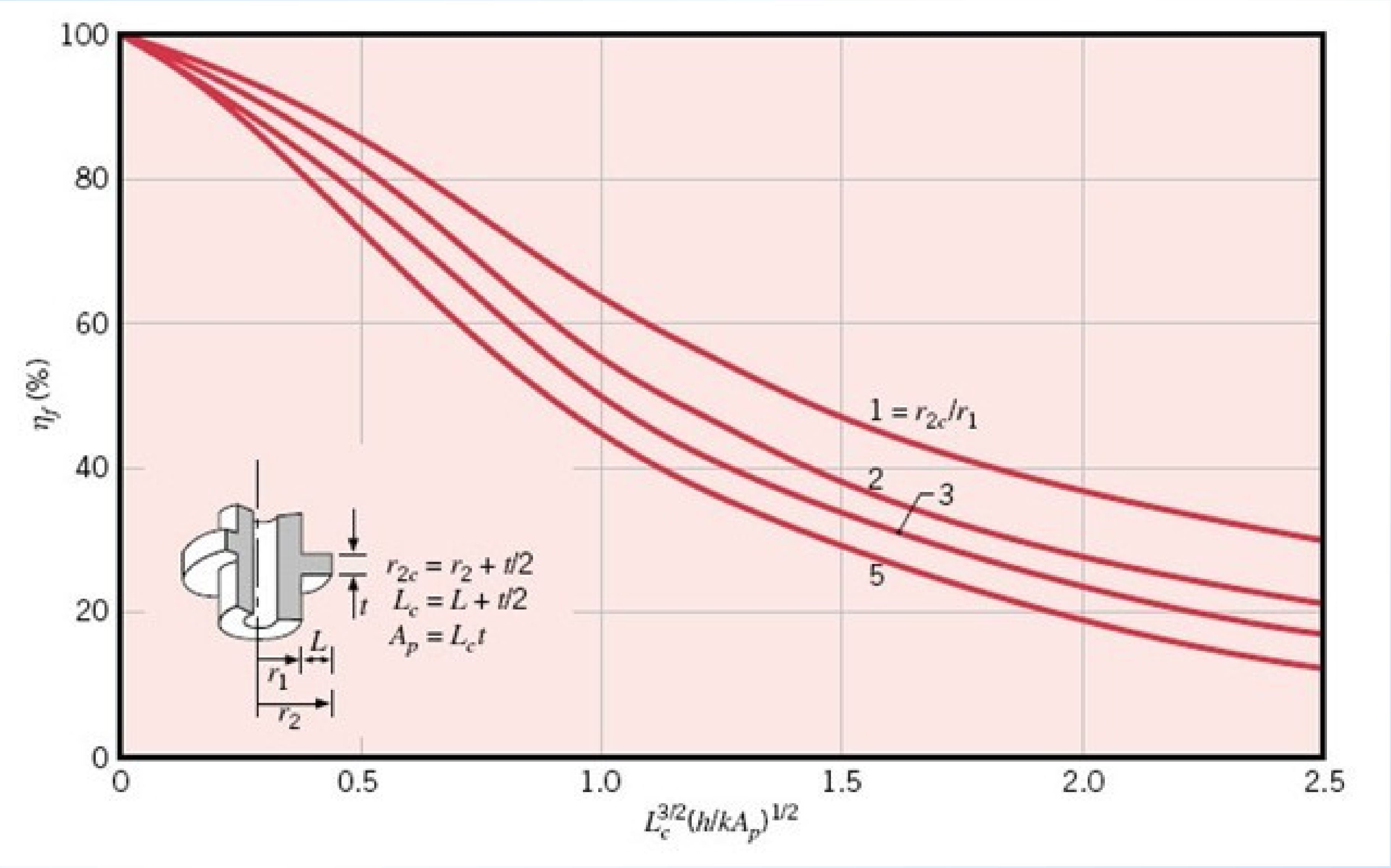
$L\sqrt{2h/kt}$ where t is the thickness of the fin at the base.

The total heat transfer from a finned surface is:

$$\dot{Q}_{total} = \dot{Q}_{fin} + \dot{Q}_{unfinned}$$
$$= \eta_{fin} A_{fin}\, h\, \theta_b + (A_{total} - A_{fin})\, h\, \theta_b$$

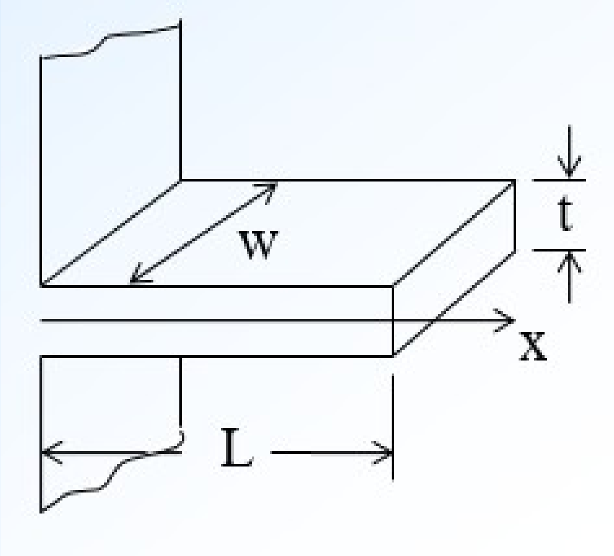
As a practical guide, the ratio (p k / A h) should be much larger than unity to justify the use of fins. Although the surface area of heat transfer is increased, the thermal resistance over the portion of the surface where the fins are attached is also increased.


100
80
60
40
20
0
0
0.5
1.0
1.5
2.0
2.5
η_f (%)
$L_c^{3/2}(h/kA_p)^{1/2}$
$y \sim x^2$
$L_c = L$
$A_p = Lt/3$
$L_c = L + t/2$
$A_p = L_c t$
$y \sim x$
$L_c = L$
$A_p = Lt/2$
L
t/2
x
y



100
80
60
40
20
0
0
0.5
1.0
1.5
2.0
2.5
η_f (%)
$L_c^{3/2}(h/kA_p)^{1/2}$
$1 = r_{2c}/r_1$
2
3
5
$r_{2c} = r_2 + t/2$
$L_c = L + t/2$
$A_p = L_c t$
L
t
r_1
r_2

Example 17



Compare η_{fin} of a plate fin of length L = 1.5 cm, thickness t = 0.2 cm, for the following cases:

a) Fin material is Al (k = 207.64 W/m.K) and h = 283.9 $W/m^2.K$

b) Fin material is steel (k = 41.5 W/m.K) and h = 510.9 $W/m^2.K$

Assume negligible heat loss from the tip.

$$\eta_{fin} = \frac{\tanh(m\,L)}{m\,L} \qquad m = \sqrt{\frac{p\,h}{A_c\,k}} \cong \sqrt{\frac{2\,h}{k\,t}} \qquad m\,L \cong L\sqrt{\frac{2\,h}{k\,t}}$$

$$\frac{p}{A_c} = \frac{2\,w + 2\,t}{w\,t} = \frac{2}{t}\left(1 + \frac{t}{w}\right) \cong \frac{2}{t} \qquad \text{if } w >> t \quad \frac{t}{w} \cong 0 \text{ besides } 1$$

(a) $$\eta_{fin} = \frac{\tanh(m\,L)}{m\,L} = \frac{\tanh\left[(0.015)\sqrt{\frac{(2)\,(283.9)}{(207.64)\,(0.002)}}\right]}{(0.015)\sqrt{\frac{(2)\,(283.9)}{(207.64)\,(0.002)}}} = 0.91$$

(b) $$\eta_{fin} = \frac{\tanh(m\,L)}{m\,L} = \frac{\tanh\left[(0.015)\sqrt{\frac{(2)\,(510.9)}{(41.5)\,(0.002)}}\right]}{(0.015)\sqrt{\frac{(2)\,(510.9)}{(41.5)\,(0.002)}}} = 0.56$$

Solve the same problem using the fin-efficiency Figure.

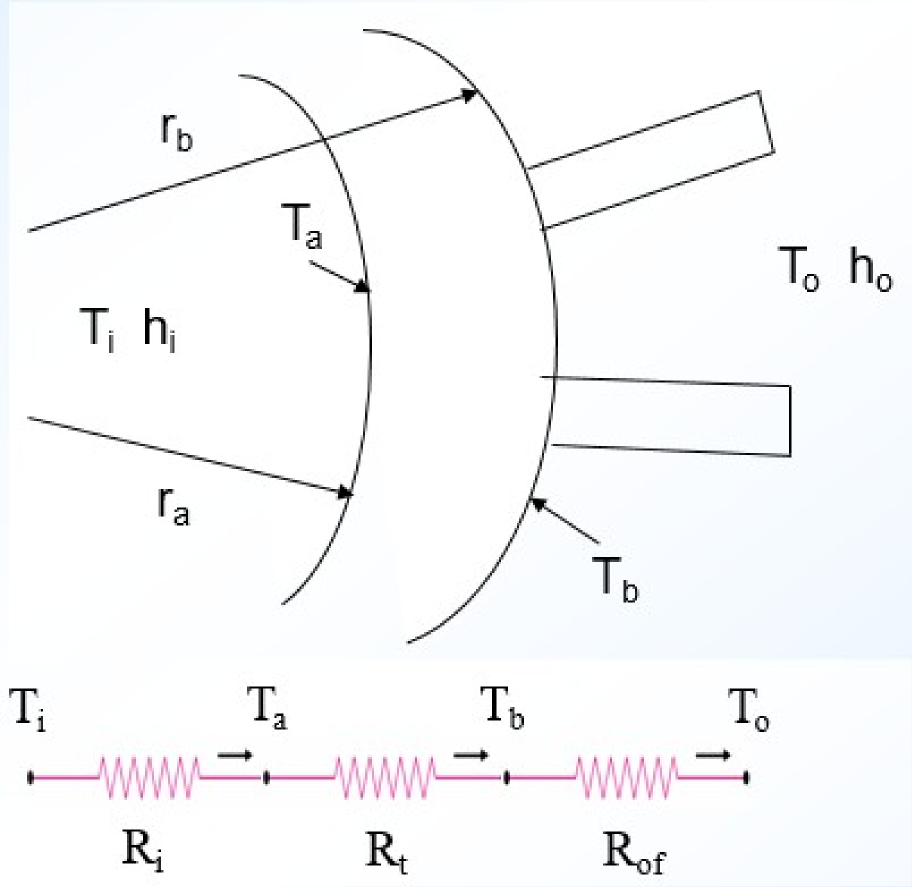
Example 18

Longitudinal thin fins are attached on the outer surface of a tube of inside radius r_a, outside radius r_b, and legth L. The hot and cold fluids flowing inside and outside the tube have mean temperatures T_i and T_o, and heat transfer coefficients h_i and h_o, respectively.

The total heat transfer area on the outer surface of the tube, including the surface areas of the fins and the unfinned portion of the tube, is A_{tot} m^2 and the ratio of the fin surface area A_{fin} to the total heat transfer are A_{tot} is β. The fin efficiency η_{fin} and the thermal conductivity k of the tube material are given.

a) Derive an expression for the heat transfer rate Q_{fin} through the finned tube.

b) Compare Q_{fin} with the heat transfer rate Q_b for the case with no fins on the tube.



The thermal resistance concept can be used as shown in the Figure.

R_{of} is the thermal resitance of the outside flow including the effects of the longitudinal fins.

Total heat transfer rate from the finned surface:

$$\dot{Q}_{fin} = \eta_{fin} A_{fin} h_o \theta_o + (A_{tot} - A_{fin}) h_o \theta_o$$

$$\dot{Q}_{fin} = \eta_{fin} \beta + (1 - \beta) A_{tot} h_o \theta_o \quad \text{if} \quad \beta = \frac{A_{fin}}{A_{tot}}$$

Define area-weighted fin efficicency: $\eta'_{fin} = \eta_{fin} \beta + (1 - \beta)$

$$\dot{Q}_{fin} = \eta'_{fin} A_{tot} h_o (T_b - T_o) = \frac{T_b - T_o}{R_{of}}$$

Therefore: $R_{of} = \dfrac{1}{\eta'_{fin} A_{tot} h_o}$

Using the known temperatutes, T_i and T_o, the total heat transfer rate from the finned surface becomes:

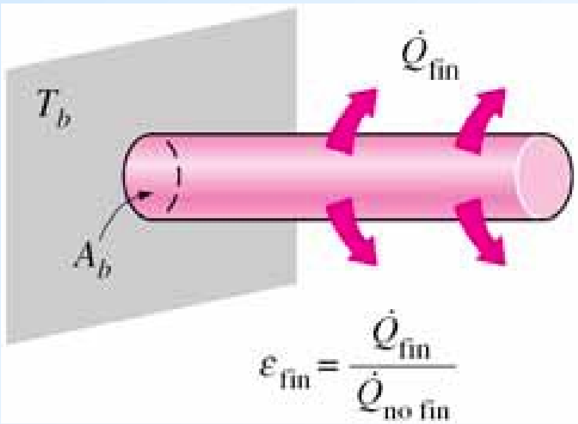
$$\dot{Q}_{fin} = \frac{T_i - T_o}{R_i + R_t + R_{of}}$$

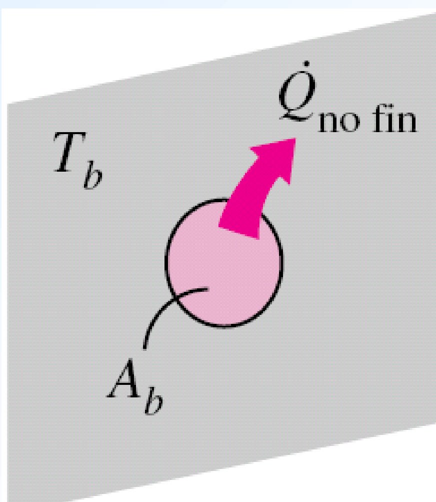
$$R_i = \frac{1}{2\,\pi\, r_a\, L\, h_i} \quad \text{and} \quad R_t = \frac{1}{2\,\pi\, L\, k} \ln\left(\frac{r_b}{r_a}\right)$$

(b) If there are no fins: $\dot{Q}_b = \dfrac{T_i - T_o}{R_i + R_t + R_o}$ where $R_o = \dfrac{1}{2\,\pi\, r_b\, L\, h_o}$

The ratio is: $\dfrac{\dot{Q}_{fin}}{\dot{Q}_b} = \dfrac{R_i + R_t + R_o}{R_i + R_t + R_{of}}$

3.11 Fin Effectiveness





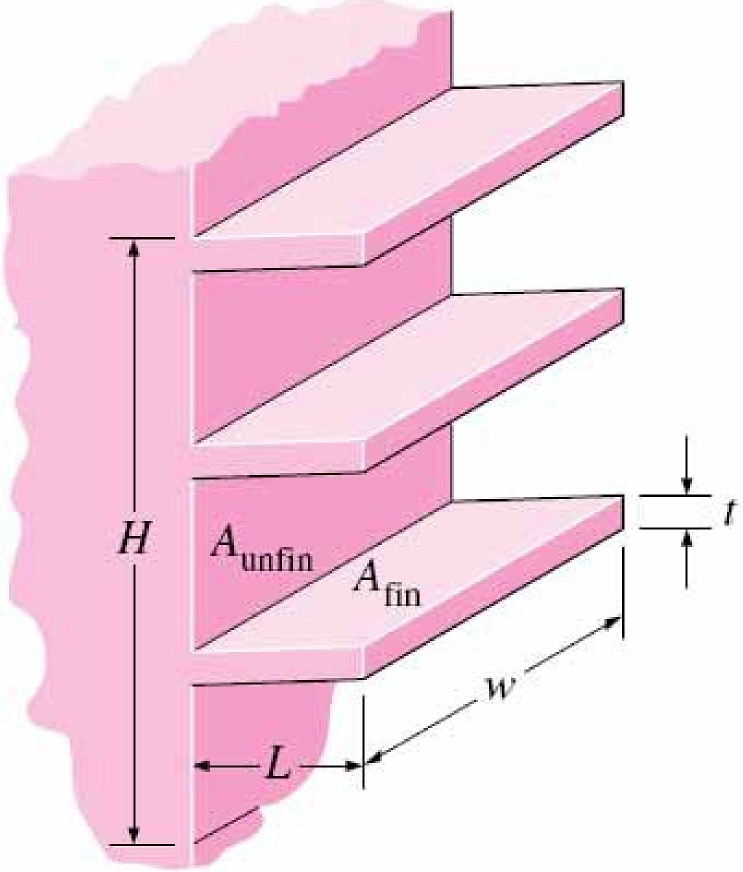
The performance of the fins is judged on the basis of the enhancement in heat transfer relative to the no-fin case.

The performance of fins is expressed in terms of the *fin effectiveness* ε_{fin} defined as

$$\varepsilon_{fin} = \frac{\dot{Q}_{fin}}{\dot{Q}_{no\ fin}} = \frac{\dot{Q}_{fin}}{h\, A_b\, (T_b - T_\infty)}$$

Remarks Regarding Fin Effectiveness

- The *thermal conductivity k of the fin material should* be as high as possible. It is no coincidence that fins are made from metals.
- The ratio of the *perimeter to the cross-sectional area* of the fin p / A_c *should be as high as possible.*
- The use of fins is *most effective in applications* involving a *low convection heat transfer coefficient.*
- Hence, the use of fins is more easily justified when the medium is a *gas instead of a liquid, and the heat* transfer is by *natural convection instead of by forced* convection.



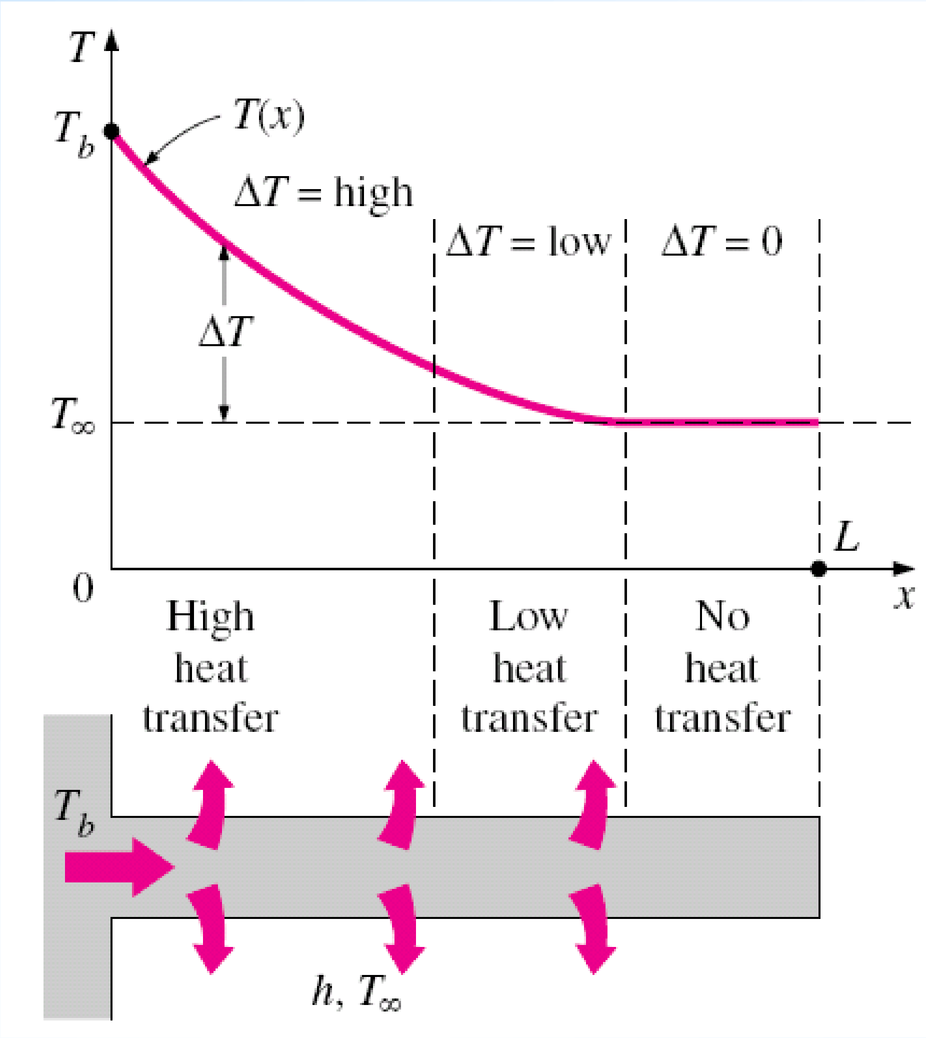
$$A_{\text{no fin}} = w \times H$$
$$A_{\text{unfin}} = w \times H - 3 \times (t \times w)$$
$$A_{\text{fin}} = 2 \times L \times w + t \times w$$
$$\cong 2 \times L \times w \text{ (one fin)}$$

Overall Effectiveness

An **overall effectiveness for a** finned surface is defined as the ratio of the total heat transfer from the finned surface to the heat transfer from the same surface if there were no fins.

$$\varepsilon_{fin,overall} = \frac{\dot{Q}_{fin}}{\dot{Q}_{no\ fin}} = \frac{h\left(A_{unfin} + \eta_{fin}\, A_{fin}\right)}{h\, A_{no\ fin}}$$

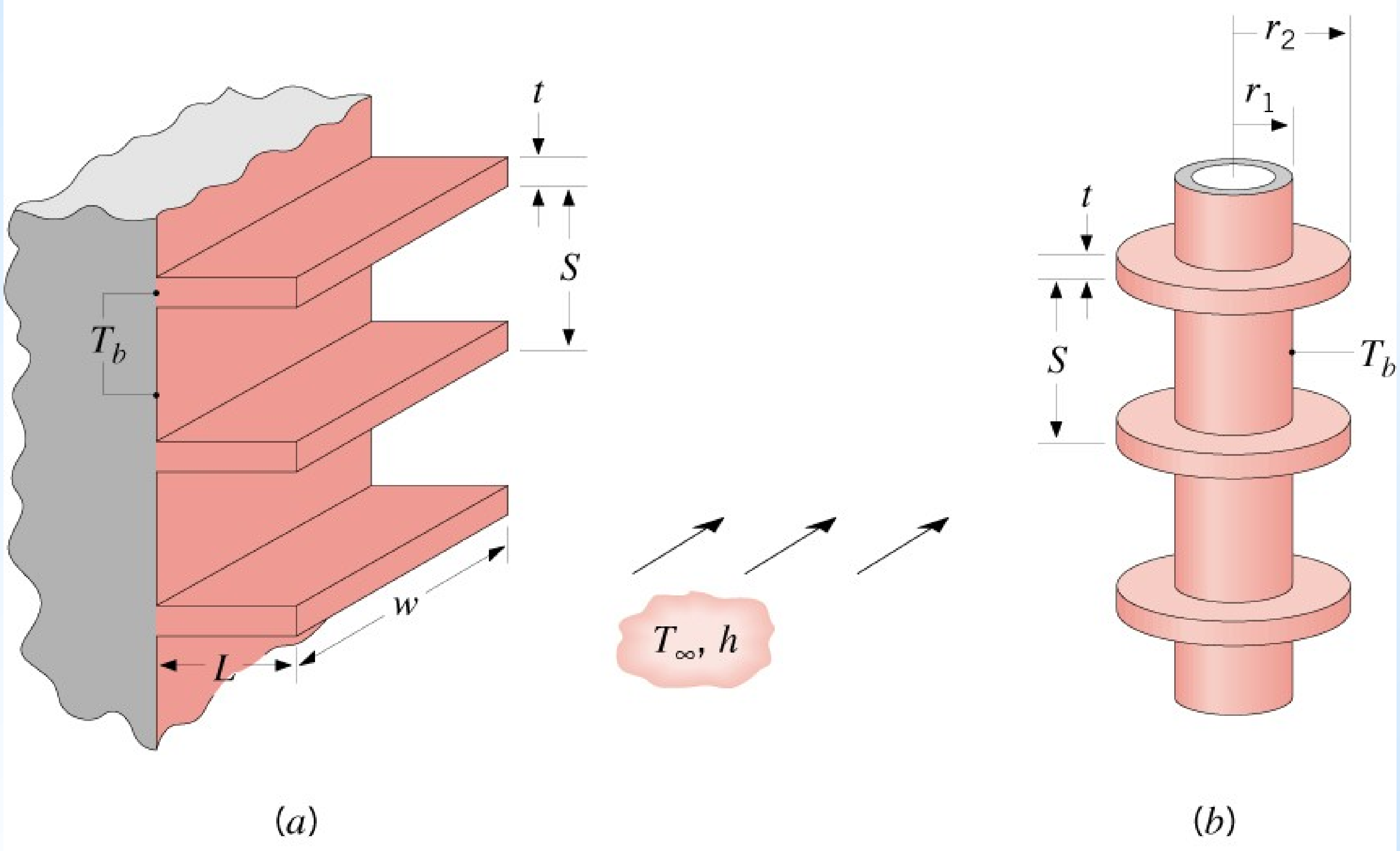
Proper Length of a Fin



An important step in the design of a fin is the determination of the appropriate length of the fin once the fin material and the fin cross section are specified.

The temperature drops along the fin exponentially and asymptotically approaches the ambient temperature at some length.

Fin Arrays



Total surface area: $A_t = N\, A_f + A_b$

N: Number of fins

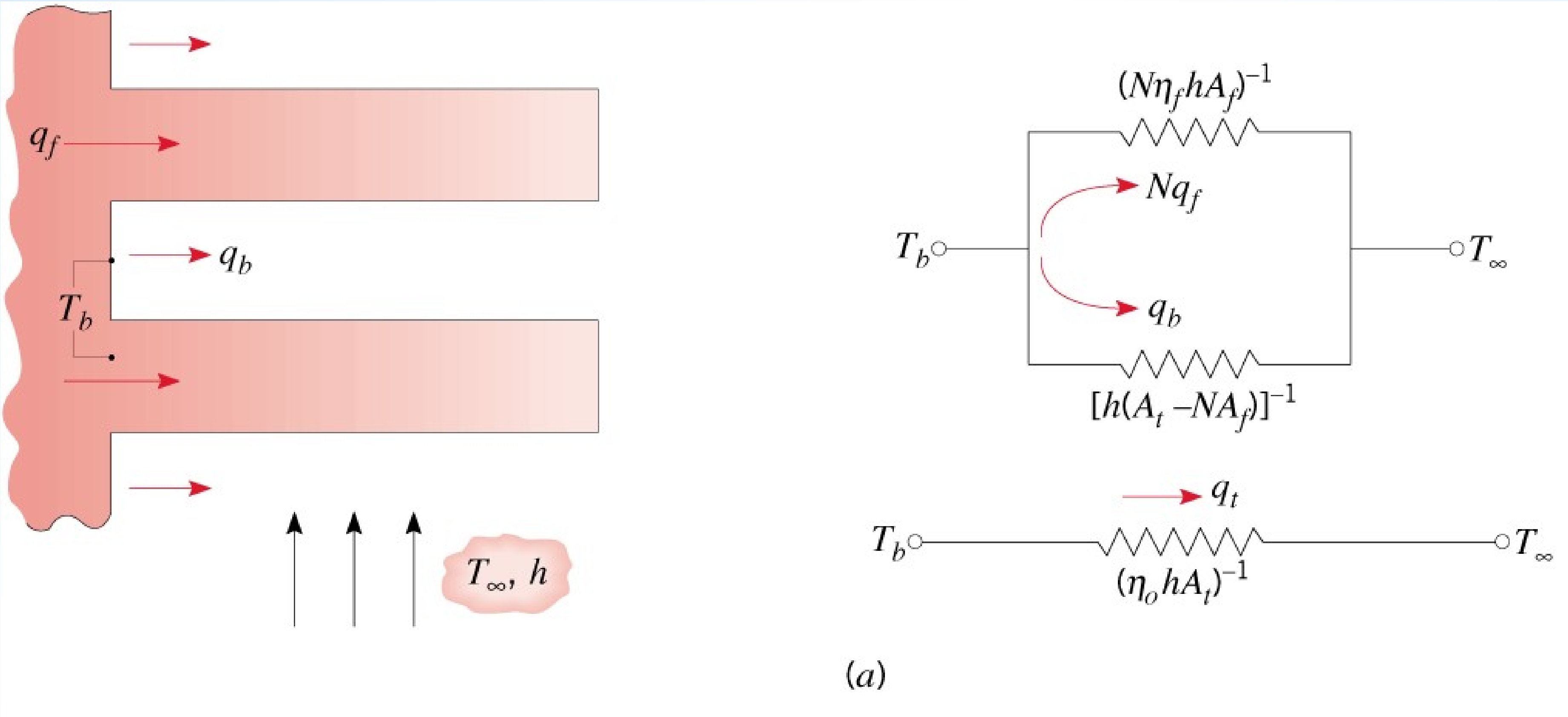
A_b: Area of exposed base (*prime* surface)

Total heat flow rate: $$\dot{Q}_t = N\, \eta_f\, h\, A_f\, \theta_b + h\, A_b\, \theta_b = \eta_0\, h\, A_t\, \theta_b = \frac{\theta_b}{R_{t,0}}$$

Overall finned-surface efficiency: $$\eta_0 = 1 - \frac{N\, A_f}{A_t}\,(1 - \eta_f)$$

Overall surface resistance: $$R_{t,0} = \frac{\theta_b}{Q_t} = \frac{1}{\eta_0\, h\, A_t}$$

Equivalent thermal circuit



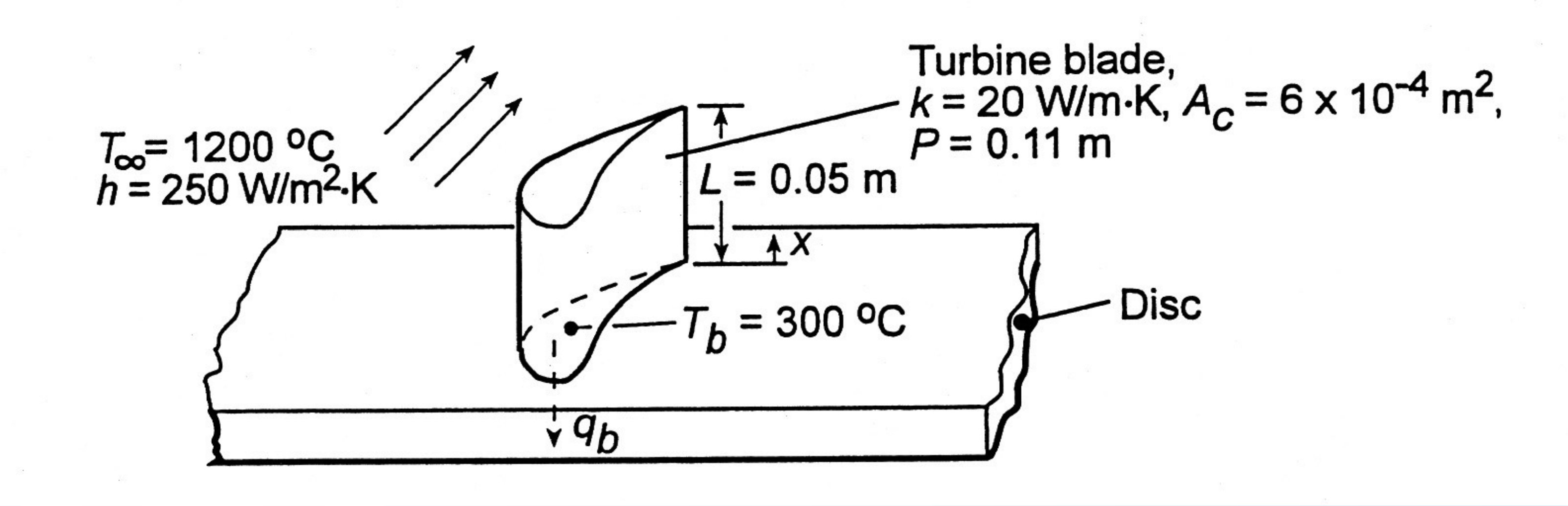
Example 19

For the turbine blade and operating conditions shown in the figure,

a) Determine whether blade temperatures are less than the maximum allowable value (1050 °C) for the prescribed operating conditions;

b) Find the heat loss from the blade.

Assume adiabatic tip.



Assumptions:

(1) One-dimensional, steady-state conduction in the blade, (2) Constant k,

(3) Adiabatic blade tip, (4) Negligible radiation.

Analysis: Conditions in the blade are determined by Case B of Table 3.4.

(a)

With the maximum temperature existing at x = L, Eq. 3.75 yields

$$\frac{T(L) - T_\infty}{T_b - T_\infty} = \frac{1}{\cosh(mL)} \qquad m = \sqrt{\frac{h\,p}{k\,A_c}} = \sqrt{\frac{(250)\,(0.11)}{(20)\,(6\;10^{-4})}} = 47.87\ \text{m}^{-1}$$

$$m\,L = 47.87\ \text{m}^{-1} \times 0.05\ \text{m} = 2.59$$

$$\cosh(m\,L) = 5.51$$

$$T(L) = 1200 + \frac{300 - 1200}{5.51} = 1037\ ^{\circ}\mathrm{C}$$

subject to the assumption of an adiabatic tip, the operating conditions are acceptable.

(b)

$$M = \sqrt{h\,p\,k\,A_c}\ \theta_b = \sqrt{(250)\,(0.11)\,(20)\,(6\ 10^{-4})}\ (-\ 900) = -\ 517\ \mathrm{W}$$

$$Q_f = M\tanh(m\,L) = (-\ 517)\,(0.983) = -\ 508\ \mathrm{W}$$

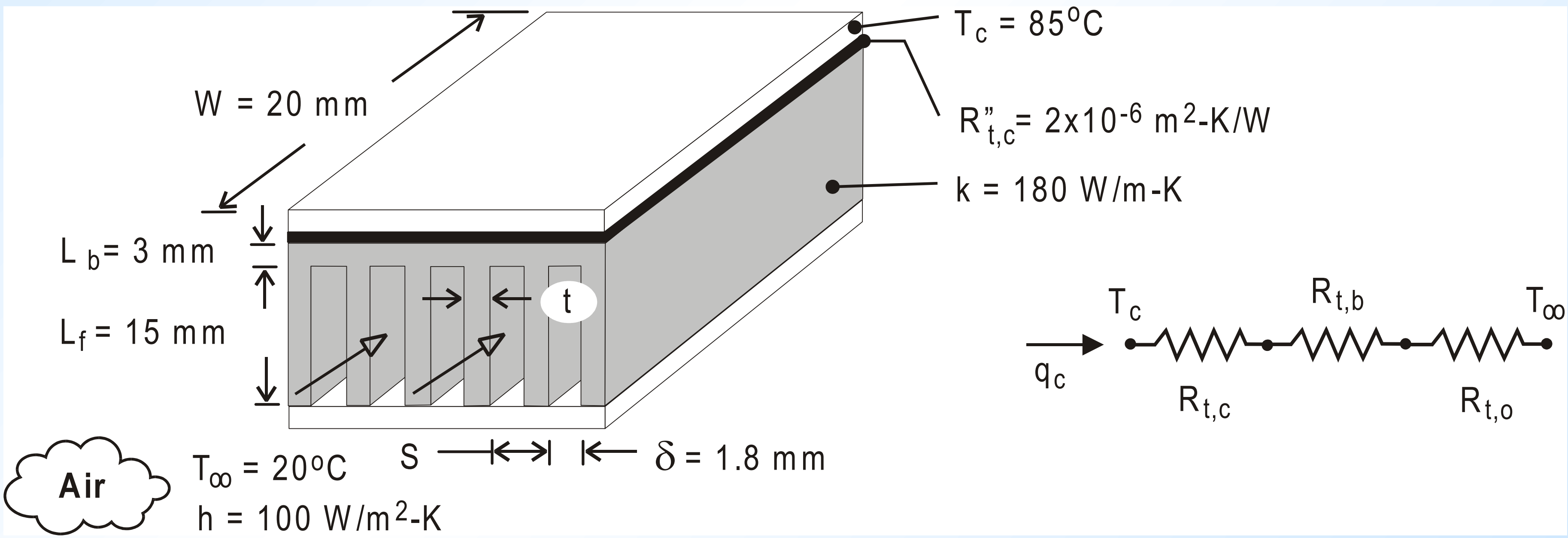
$$Q_b = -\,Q_f = 508\ \mathrm{W}$$

Comments:

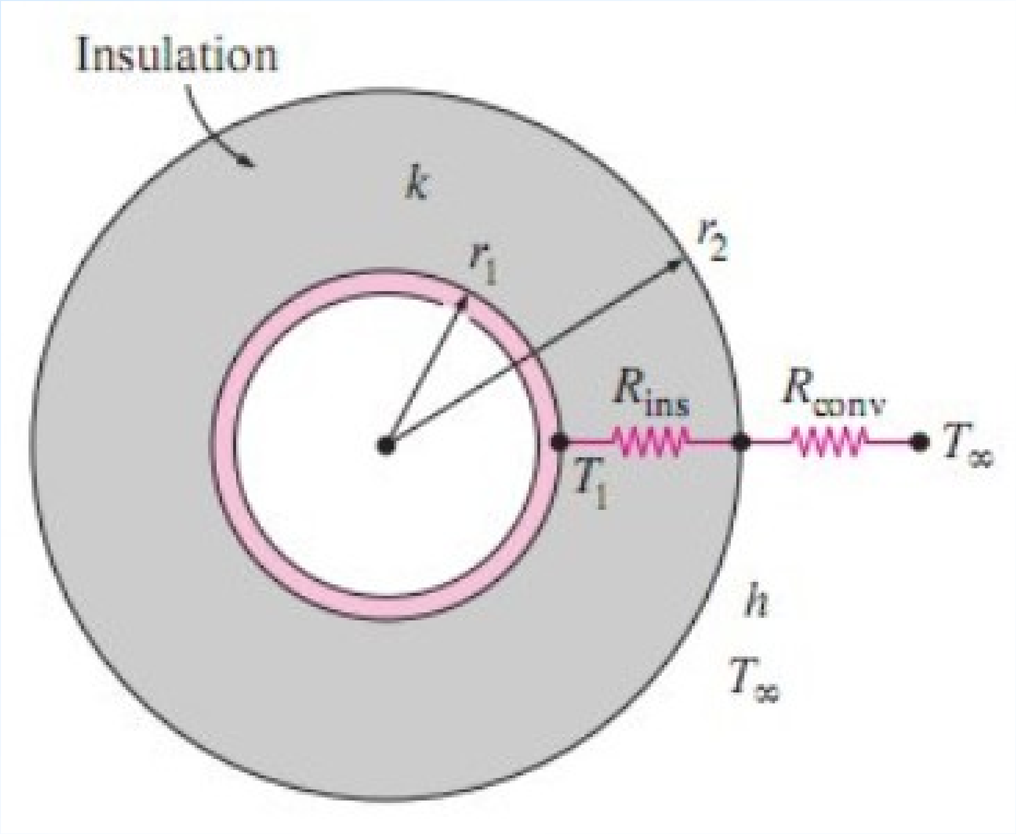
Radiation losses from the blade surface contribute to reducing the blade temperatures, but what is the effect of assuming an adiabatic tip condition? Calculate the tip temperature allowing for convection from the gas.

Example 20

Determine the maximum allowable power for a 20 mm x 20 mm electronic chip whose temperature is not to exceed 85 °C when the chip is attached to an air-cooled heat sink with N = 11 fins of prescribed dimensions.



3.12 Critical Radius of Insulation



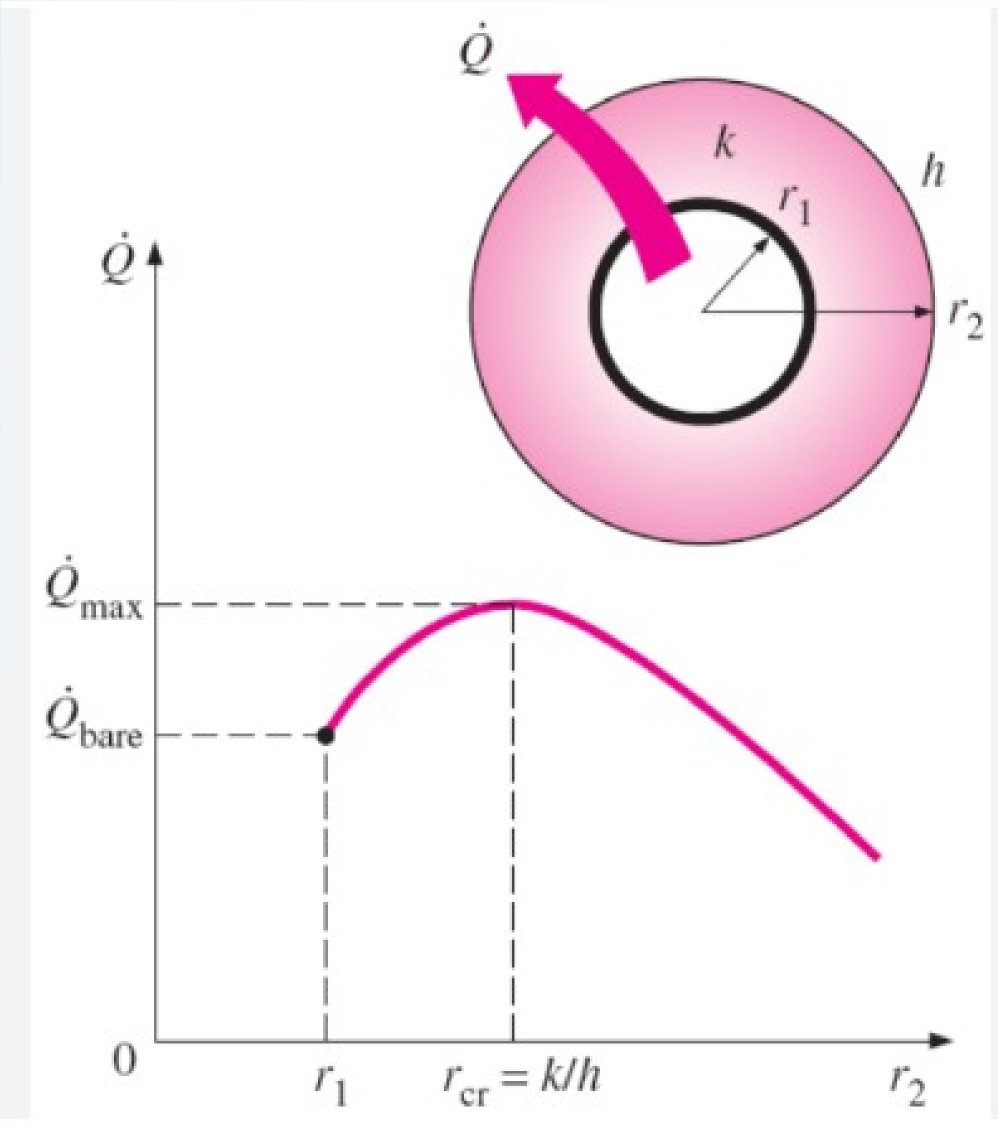
Consider the insulated cyclinder shown in the Figure. T_1 is kept constant.

Heat flow rate: $$\dot{Q} = \frac{T_1 - T_\infty}{R_{ins} + R_{conv}}$$

$$R_{ins} = \frac{1}{2\pi L k} \ln\left(\frac{r_2}{r_1}\right)$$

$$R_{conv} = \frac{1}{2\pi r_2 L h_o}$$

If r_2 increases, R_{ins} increases, but R_{conv} decreases. What happens to Q?



Q reaches a maximum value at a certain radius r_2, and this is called the critical radius of insulation, $r_2 = r_{cr}$.

Set the first derivative of Q with respect to r_2 equal to zero. The maximum occurs at r_{cr}.

$$r_{cr} = \frac{k}{h_o}$$

$$\frac{d\dot{Q}}{dr_2} = - \frac{2\pi k L (T_1 - T_\infty)}{\left[\ln\left(\frac{r_2}{r_1}\right) + \frac{k}{h_0 r_2}\right]^2} \left(\frac{1}{r_2} - \frac{k}{h_0 r_2^2}\right) = 0 \quad \Rightarrow \quad r_{cr} = \frac{k}{h_o}$$

Physical significance: Heat loss from a pipe increases with addition of insulation if pipe radius r_1 is less than r_{cr} until r_{cr} is reached. Then, it starts to decrease.

If the effect of heat loss by radiation is included, the critical radius is somewhat lowered.